Paradoxes of $\mathcal{PT}-$symmetric Quantum Mechanics

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Summary (eight paradoxes):

1. Perturbation paradox for AHO .

Contemplate $V(x) = x^2 + g\,x^4$. *We know:* $E(g)$ *is analytic. Its Taylor series is* reasonably *easy at* $g_0 = 0$. *Still, its essential singularity lies* ausgerechnet *there.*

2. Solvability paradox .

Everybody chooses $E(g)$ *as real for* $g > 0$ *(not for* $g < 0$*), not being aware that an alternative convention would work as well. The wedge choice is ambiguous, indeed. Unexpectedly, re-quantizations might exhibit non-trivial merits (QES).*

Two related features of *cubic* AHO:

3. Non-escape paradox at $\mathrm{Im}\, g \neq 0$.

The classical time-of-flight-left remains finite in $V(x) = x^2 + g\,x^3$ *with* $g > 0$. *Still, in a close parallel to a fall into a strong singularity one gets everything OK after a "counter-intuitive" re-quantization demanding* $\mathrm{Re}\, g > 0$.

4. Summability paradox in φ^3 .

Green's function is a formal power series in g^2. *Is* not *Borel summable for real* g *since it does not alternate in sign. A "unique reasonable" choice of* g *is purely imaginary, with*

$$\mathcal{L}_{int} \sim i\,\varphi^3.$$

Feature: it exhibits $\mathcal{PT}$ symmetry as if it were a "weak hermiticity".

First steps of development in field theory:

5. Fixed-point paradox in QED .

Bender and Milton re-examined the program of Johnson, Baker and Willey. Three-loop calculation seems to support the existence of a finite QED *(with $1/Z_3 \neq \infty$ at a suitable fixed point $\alpha \neq 0$) whenever one switches from real to imaginary charges in a $\mathcal{PT}-$symmetric way.*

6. Parity breaking paradox in φ^4 .

In a $\mathcal{PT}-$symmetric quantization using Feynman integrals in D dimensions Bender and Milton evaluated $\langle\varphi\rangle \neq 0$ non-perturbatively. The parity symmetry is, counter-intuitively, spontaneously broken.

Final *"take home"* **about** $\mathcal{PT}$ **world:**

7. Unexpected user-friendliness .

Anharmonic oscillators $V(x) \sim \sum_{k=1}^{3} g_k (ix)^k$ *are a fresh and most elementary testing ground for perturbation theory, especially in the strong-coupling regime. They form a new promising methodical laboratory for various analytic or algebraic and numerical techniques.*

8. Unexpected universality .

$\mathcal{PT}-$*symmetric quantum mechanics is full of positive surprizes: Oscillators* $V(x) \sim (ix)^\delta$ *are "smooth" in* δ*, forming non-equivalent classes. One may move up to the* $\delta \to \infty$ *limiting cases. Some of them return/live on the real axis of coordinates. Open to further generalizations.*