Polynomial oscillators in large dimensions

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- quasi-quasi-exact systems of Magyari: polynomial wave functions, $q > 1$
- growing degree N (the key shortcoming: energies become numerical)
- large dimensions D (key result: multiple, decisive, mysterious simplifications)
- perturbation corrections: $1/D$ series (boring but user-friendly mathematics)
- $q \geq 6$: *hic sunt leones*.

Exactly solvable oscillators

Harmonic oscillator and the like

$$E = E_{n,L} = \omega\,(2n + \ell_L + 3/2)$$

All bound states are elementary

$$\psi_{n,L}(r) = r^{\ell+1}\,e^{-\lambda(r)}\,\sum_{m=0}^{N-1} h_m\,r^{2m}$$

Ground state is Gaussian

$$\lambda(r) = \omega\,r^2/2 > 0$$

Re-construction of the potential

$$\left[-\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2} + \omega^2 r^2\right]\psi(r) = E\,\psi(r),$$

$$\ell = \ell_L = L + \frac{D-3}{2}$$

Quasi-exactly solvable oscillators

Illustration: Sextic oscillators

$$V^{(sextic)}(r) = g_0\, r^2 + g_1\, r^4 + g_2\, r^6$$

Similarity # 1: ground state is next to Gaussian

$$\lambda(r) = \frac{1}{2}\alpha_0 r^2 + \frac{1}{4}\alpha_1 r^4$$

Similarity # 2: V is next to HO

$$\omega \longrightarrow W(r) = \alpha_0 + \alpha_1 r^2$$

$$V^{(sextic)}(r) = U(r) + r^2 W^2(r)$$

$$g_2 = \alpha_1^2 > 0, \quad g_1 = 2\,\alpha_0\alpha_1, \quad g_0 = 2\,\alpha_0^2 + G_0\,.$$

Many bound states are elementary: specific constraint

$$G_0 = -\alpha_0^2 - \alpha_1(4N + 2\ell + 1), \qquad N \geq 1\ .$$

Fully general case

Potentials

$$V^{(q)}(r) = U^{(q)}(r) + r^2 [W^{(q)}(r)]^2$$
$$U^{(q)}(r) = G_0 r^2 + G_1 r^4 + \ldots + G_{q-1} r^{2q}$$
$$W^{(q)}(r) = \alpha_0 + \alpha_1 r^2 + \ldots + \alpha_q\, r^{2q}$$

Polynomials

$$V^{[q]}(r) = g_0\, r^2 + g_1\, r^4 + \ldots + g_{2q}\, r^{4q+2}$$

$$\{g_0, \ldots, g_{2q}\} \Longleftrightarrow \{G_0, \ldots, G_{q-1}, \alpha_0, \ldots, \alpha_q\}$$

Polynomial Gaussians

$$\lambda^{(q)}(r) = \frac{1}{2}\alpha_0 r^2 + \frac{1}{4}\alpha_1 r^4 + \ldots + \frac{1}{2q+2}\alpha_q r^{2q+2}\,.$$

$$\psi(r) = \sum_{n=0}^{N-1} h_n^{(N)}\, r^{2n+\ell+1} \exp\left[-\lambda^{(q)}(r)\right]$$

Terminator

$$\hat{Q}^{[N]}\,\vec{h}^{(N)} = 0$$

with matrix

$$\begin{pmatrix}
B_0 & C_0 & & & & & & \\
A_1^{(1)} & B_1 & C_1 & & & & & \\
\vdots & & \ddots & \ddots & & & & \\
A_q^{(q)} & \ldots & A_q^{(1)} & B_q & C_q & & & \\
 & \ddots & & & \ddots & \ddots & & \\
 & & A_{N-2}^{(q)} & \ldots & A_{N-2}^{(1)} & B_{N-2} & C_{N-2} \\
 & & & A_{N-1}^{(q)} & \ldots & A_{N-1}^{(1)} & B_{N-1} \\
 & & & & \ddots & \vdots & \vdots \\
 & & & & & A_{N+q-2}^{(q)} & A_{N+q-2}^{(q-1)} \\
 & & & & & & A_{N+q-1}^{(q)}
\end{pmatrix}.$$

and elements

$$C_n = (2n+2)\,(2n+2\ell+3), \qquad B_n = E - \alpha_0\,(4n+2\ell+3)$$
$$A_n^{(1)} = -\alpha_1\,(4n+2\ell+1) + \alpha_0^2 - g_0, \qquad A_n^{(2)} = -\alpha_2\,(4n+2\ell-1) + 2\alpha_0\alpha_1 - g_1,$$
$$\ldots,$$
$$A_n^{(q)} = -\alpha_q\,(4n+2\ell+3-2q) + (\alpha_0\alpha_{q-1} + \alpha_1\alpha_{q-2} + \ldots + \alpha_{q-1}\alpha_0) - g_{q-1},$$
$$n = 0, 1, \ldots \ .$$

Separate solvability condition

$$g_{q-1} = -\alpha_q\,(4n+2\ell+3-2q) + (\alpha_0\alpha_{q-1} + \alpha_1\alpha_{q-2} + \ldots + \alpha_{q-1}\alpha_0)\,.$$

$$A_n^{(q)} = 4\,\gamma\,(N+q-n-1), \qquad A_{N+q-1}^{(q)} = 0$$

$D \gg 1$ asymptotic domain

$$
\begin{gathered}
C_n = (2n+2)\,(2n+2L+D), \\
B_n = -g_{-1} - \alpha_0\,(4n+2L+D), \\
A_n^{(k)} = -g_{k-1} - \alpha_k\,(4n+2L+D-2k) + D, \\
D = (\alpha_0\alpha_{k-1} + \ldots + \alpha_{k-1}\alpha_0)\,, \\
k = 1, 2, \ldots, q-1, \qquad n = 0, 1, \ldots, N+q-2\ .
\end{gathered}
$$

Re-scaling

$$
h_n^{(N)} = p_n/\mu^n\ , \qquad \frac{2D}{\mu} = \tau = 4\,\gamma\,\mu^q.
$$

$$
\mu = \left(\frac{D}{2\gamma}\right)^{1/(q+1)}, \quad \tau = \left(2^{q+2}\,D^q\,\gamma\right)^{1/(q+1)}.
$$

$$
\{g_{-1}, g_0, \ldots, g_{q-2}\} \to \{s_1, s_2, \ldots, s_q\}
$$

$$
g_{k-2} = -\alpha_{k-1} D - \frac{\tau}{\mu^{k-1}}\, s_k, \qquad k = 1, 2, \ldots, q\ .
$$

Standard approach

$$\frac{\ell(\ell+1)}{r^2}+\omega^2 r^2 = 2\omega^2 R^2+4\,\omega^2(r-R)^2+\ldots$$

$$R=\left[\ell(\ell+1)/\omega^2\right]^{1/4}\gg 1$$

$$\psi_0\sim e^{-\omega\,(r-R)^2},\qquad \psi_1\sim(r-R)\,e^{-\omega\,(r-R)^2},\ \ \ldots,\qquad r\approx R$$

$$E_0=2\omega^2 R^2+2\omega+\ldots,\qquad E_1=2\omega^2 R^2+6\omega+\ldots,\ \ \ldots\,.$$

Magyari's matrix

$$\left(\begin{array}{cccccc}
s_1 & 1 & & & & \\
s_2 & s_1 & 2 & & & \\
\vdots & & \ddots & \ddots & & \\
s_q & \vdots & & s_1 & N-2 & \\
N-1 & s_q & & & s_1 & N-1 \\
 & N-2 & s_q & & \vdots & s_1 \\
 & & \ddots & \ddots & & \vdots \\
 & & & 2 & s_q & s_{q-1} \\
 & & & & 1 & s_q
\end{array}\right)$$

Harmonic oscillator: $q = 0$

$$\begin{aligned} V^{(q=0,k=1)}(r) &= V^{(HO)}(r) = r^2 \\ V^{(q=0,k=2)}(r) &= V^{(Coulomb)}(r) = ar^{-1} \\ V^{(Morse)}(r) &= [a\,exp(-\alpha\,x) - b]^2 \end{aligned}$$

Recurrences

$$\begin{pmatrix} N-1 & 1 & & & \\ & \ddots & \ddots & & \\ & & 2 & N-2 & \\ & & & 1 & N-1 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{N-2} \\ p_{N-1} \end{pmatrix} = 0$$

Solutions

$$p_n = (-1)^n\, p_0 \begin{pmatrix} N-1 \\ n \end{pmatrix} \ .$$

$$\psi_{n,L}(r) = r^{\ell+1} \left(1 - \frac{r^2}{\mu}\right)^n e^{-\lambda\, r^2}\,, \qquad n = 0, 1, \ldots\ .$$

$q = 1$ QES oscillators

$$
\begin{aligned}
V^{(q=1,k=1)}(r) &= ar^2 + br^4 + r^6 \\
V^{(q=1,k=2)}(r) &= ar^{-1} + br + r^2 \\
V^{(q=1,k=3)}(r) &= ar^{-4/3} + br^{-2/3} + r^{2/3} \\
V^{(q=1,k=4)}(r) &= ar^{-3/2} + br^{-1} + cr^{-1/2} \ .
\end{aligned}
$$

Magyari's diagonalization

$$
\begin{pmatrix}
s & 1 & & & & \\
N-1 & s & 2 & & & \\
 & N-2 & s & 3 & & \\
 & & \ddots & \ddots & \ddots & \\
 & & & 2 & s & N-1 \\
 & & & & 1 & s
\end{pmatrix}
\begin{pmatrix}
p_0 \\ p_1 \\ \vdots \\ p_{N-2} \\ p_{N-1}
\end{pmatrix} = 0
$$

Magyari's secular determinants

$$
\begin{aligned}
s^3 - 4\,s = 0, &\qquad N = 3, \\
s^4 - 10\,s^2 + 9 = 0, &\qquad N = 4, \quad etc
\end{aligned}
$$

Energies

$$s = s^{(j)} = -N-1+2j, \qquad j = 1, 2, \ldots, N.$$

Wave functions

Coefficients

$$\begin{gathered}
p_0^{(1)} = 1, \qquad\qquad N = 1, \\
p_0^{(1)} = p_1^{(1)} = p_0^{(2)} = -p_1^{(2)} = 1, \qquad\quad N = 2, \\
p_0^{(1)} = p_2^{(1)} = p_0^{(2)} = -p_2^{(2)} = p_0^{(3)} = p_2^{(3)} = 1, \\
p_1^{(1)} = -p_1^{(3)} = 2, \quad p_1^{(2)} = 0, \quad N = 3,
\end{gathered}$$

Polynomials

$$\psi^{(j)}(r) = r^{\ell+1} \left(1 + \frac{r^2}{\mu}\right)^{N-j} \left(1 - \frac{r^2}{\mu}\right)^{j-1} \times$$
$$\times \exp\left(-\frac{1}{2}\alpha_0 r^2 - \frac{1}{4}\alpha_1 r^4\right) \qquad\qquad j = 1, 2, \ldots, N\,.$$

$q=2$: potentials

$$
\begin{gathered}
V^{(q=2,k=1)}(r) = ar^2 + br^4 + cr^6 + dr^8 + r^{10} \\
V^{(q=2,k=2)}(r) = ar^{-1} + br + cr^2 + dr^3 + r^4 \\
V^{(q=2,k=3)}(r) = ar^{-4/3} + br^{-2/3} + cr^{2/3} + dr^{4/3} + r^2 \\
V^{(q=2,k=4)}(r) = ar^{-3/2} + br^{-1} + cr^{-1/2} + dr^{1/2} + r \\
V^{(q=2,k=5)}(r) = ar^{-8/5} + br^{-6/5} + cr^{-4/5} + dr^{-2/5} + r^{2/5} \\
V^{(q=2,k=6)}(r) = ar^{-5/3} + br^{-4/3} + cr^{-1} + dr^{-2/3} + fr^{-1/3}
\end{gathered}
$$

$N = (1, 2,)\, 3, 4, \ldots$

$$
\begin{pmatrix}
s & 1 & & & & & & \\
t & s & 2 & & & & & \\
N-1 & t & s & 3 & & & & \\
 & N-2 & t & s & 4 & & & \\
 & & \ddots & \ddots & \ddots & \ddots & & \\
 & & & 3 & t & s & N-1 \\
 & & & & 2 & t & s \\
 & & & & & 1 & t
\end{pmatrix}
\begin{pmatrix}
p_0 \\ p_1 \\ \vdots \\ p_{N-2} \\ p_{N-1}
\end{pmatrix} = 0 \ .
$$

$$
\begin{gathered}
s^6 - 7\, s^3 - 8 = 0, \\
s^{(1)} = 2, \quad s^{(2)} = -1, \qquad N = 3, \quad q = 2
\end{gathered}
$$

$$
\begin{gathered}
s^{10} - 27\, s^7 + 27\, s^4 - 729\, s = 0 \\
s^{(1)} = 3, \quad s^{(2)} = 0, \qquad N = 4, \quad q = 2.
\end{gathered}
$$

General solution

$$
s^{(j)} = N + 2 - 3j, \qquad j = 1, 2, \ldots, j_{max}, \qquad j_{max} = entier \left[\frac{N+1}{2} \right] \ .
$$

$q = 3$

Potentials

$$V^{(q=3,k=1)}(r) = a\,r^2 + b\,r^4 + \ldots + g\,r^{14}$$
$$V^{(q=3,k=2)}(r) = \frac{e}{r} + a\,r + b\,r^2 + \ldots + f\,r^6$$

etc

Pentadiagonal matrices

$$\begin{pmatrix}
r & 1 & & & & & & & \\
s & r & 2 & & & & & & \\
t & s & r & 3 & & & & & \\
N-1 & t & s & r & 4 & & & & \\
 & N-2 & t & s & r & 5 & & & \\
 & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\
 & & & 4 & t & s & r & N-1 \\
 & & & & 3 & t & s & r \\
 & & & & & 2 & t & s \\
 & & & & & & 1 & t
\end{pmatrix}$$

Step-by-step solution

$$t^9 - 12\,t^5 - 64\,t = 0, \qquad N = 3$$

$$t^{16} - 68\,t^{12} - 442\,t^8 - 50116\,t^4 + 50625 = 0, \qquad N = 4$$

$$t^{25} - 260\,t^{21} + \ldots + 10485760000\,t = 0, \qquad N = 5$$

$$t^{36} - 777\,t^{32} + \ldots - 7547691631289 0625 = 0, \qquad N = 6$$

General solution

$$s = s^{(j)} = N + 3 - 4j,$$
$$r = r^{(j,k)} = t = t^{(j,k)} = -N - 3 + 2j + 2k,$$
$$k = 1, 2, \ldots, k_{max}(j),$$
$$k_{max}(j) = N + 2 - 2j \;\; ,$$
$$j = 1, 2, \ldots, j_{max} \;\; ,$$
$$j_{max} = entier\left[\frac{N+1}{2}\right] \;\; .$$

$q = 4$

$$V(r) = r^3 + \ldots \textbf{ etc}$$

$N = 4$

$$P(x) = (x+3)\ P_{+\sqrt{5}}(x) \cdot P_{-\sqrt{5}}(x),$$

$$P_{\sqrt{5}}(x) = \left(2\,x + 1 + \sqrt{5}\right) \times F \times G$$

$$F = \left(x^2 + x + \sqrt{5}x + 4 + \sqrt{5}\right) > 0$$

$$G = \prod_{k=1}^{7} \left(2\,x^2 + \delta_k\,x + \gamma_k\right) > 0$$

k	γ_k	δ_k
1	18	$-3 + 3\sqrt{5}$
2	$3 + \sqrt{5}$	2
3	$3 + \sqrt{5}$	$-3 - \sqrt{5}$
4	$8 + 2\sqrt{5}$	$7 + \sqrt{5}$
5, 6	$8 + 2\sqrt{5}$	$-3 \pm \sqrt{5}$
7	$8 + 2\sqrt{5}$	$-3 - 3\sqrt{5}$

$q = N = 4$	
s_3	s_4
3	3
$(\sqrt{5}+1)/2$	$(-\sqrt{5}+1)/2$
$(-\sqrt{5}+1)/2$	$(\sqrt{5}+1)/2$
$N = 5$	
s_3	s_4
-1	-1
4	4
$\sqrt{5}-1$	$-\sqrt{5}-1$
$-\sqrt{5}-1$	$\sqrt{5}-1$
$(\sqrt{5}+3)/2$	$(-\sqrt{5}+3)/2$
$(-\sqrt{5}+3)/2$	$(\sqrt{5}+3)/2$
$N = 6$	
s_3	s_4
0	0
5	5
$\sqrt{5}$	$-\sqrt{5}$
$-\sqrt{5}$	$\sqrt{5}$
$(\sqrt{5}+5)/2$	$(-\sqrt{5}+5)/2$
$(-\sqrt{5}+5)/2$	$(\sqrt{5}+5)/2$

$q = 5, \quad N = 6$

roots		
s_3	s_4	s_5
-5	5	-5
-3	5	-3
-1	5	-1
1	5	1
3	5	3
5	5	5
-5	-1	1
-3	-1	3
-1	-1	-1
1	-1	1
3	-1	-3
5	-1	-1
-5	2	-2
-3	2	0
-1	2	-4
-1	2	2
1	2	-2
1	2	4
3	2	0
5	2	2

$N=6$

$$P(x) = x^{91} - 16120\,x^{85} + \ldots = \prod_{j=1}^{4} P_j(x)$$

$$P_1^{(6)} = x \prod_{m=1}^{5} \left(x^2 - m^2\right) \qquad \textit{can vanish}$$

$$P_2^{(6)} = \prod_{k=1}^{2} \prod_{m=-1}^{1} \left(x^2 + 3k\,m\,x + 3k^2\right) > 0$$

$$P_3^{(6)} = \prod_{k=1}^{5} \left(x^2 - k\,x + k^2\right)\left(x^2 + k\,x + k^2\right) > 0$$

$$P_4^{(6)} = \prod_{k=1}^{12} \left(x^2 - b_k\,x + c_k\right)\left(x^2 + b_k\,x + c_k\right) > 0$$

k	c_k	b_k
$1-3$	7	$1,4,5$
$4-6$	13	$2,5,7$
$7-9$	19	$1,7,8$
$10-12$	21	$3,6,9$

$N = 7$

$$P(x) = x^{127} - 60071\, x^{121} + \ldots + c\, x \cdot 10^6 = 0$$

$$F_1^{(7)} = \left(x^2 - 6^2\right)$$

$$F_2^{(7)} = \left(x^2 - 9\,x + 27\right)\left(x^2 + 27\right)\left(x^2 + 9\,x + 27\right)$$

$$F_3^{(7)} = \left(x^2 - 6\,x + 36\right)\left(x^2 + 6\,x + 36\right)$$

$$F_4^{(7)} = \prod_{k=1}^{6} \left(x^2 - f_k\,x + g_k\right)\left(x^2 + f_k\,x + g_k\right)$$

k	f_k	g_k
$1-3$	28	$2, 8, 10$
$4-6$	31	$4, 7, 11$

$1/D \ll 1$ perturbation series

$q = 1$: The case of the sextic oscillations

$$H(\lambda) = H^{(0)}+\lambda\, H^{(1)}+\lambda^2 H^{(2)}, \qquad \lambda = 1/\sqrt{D}.$$

$$\left(H^{(1)}\right)_{nn} = \frac{\beta}{\sqrt{2\gamma}}(2n+k), \qquad n = 0, 1, \ldots, N-1,$$

$$\left(H^{(2)}\right)_{nn+1} = -(n+1)(2n+2k),$$
$$n = 0, 1, \ldots, N-2.$$

Rayleigh-Schrödinger recipe

$$\left(H^{(0)} + \lambda\, H^{(1)} + \lambda^2 H^{(2)}\right) \cdot$$
$$\cdot \left(\psi^{(0)} + \lambda\, \psi^{(1)} + \ldots + \lambda^K \psi^{(K)} + \mathcal{O}(\lambda^{K+1})\right) =$$
$$= \left(\psi^{(0)} + \ldots + \lambda^K \psi^{(K)} + \mathcal{O}(\lambda^{K+1})\right) \cdot$$
$$\cdot \left(\varepsilon^{(0)} + \ldots + \lambda^K \varepsilon^{(K)} + \mathcal{O}(\lambda^{K+1})\right).$$

Leading-order diagonalization

$$\begin{pmatrix} E-\beta D & 2D & & & \\ 4(N-1)\gamma & E-\beta D & 4D & & \\ & \ddots & \ddots & \ddots & \\ & & 8\gamma & E-\beta D & 2(N-1)D \\ & & & 4\gamma & E-\beta D \end{pmatrix}$$

$$E = \beta D - 2\sqrt{2\gamma D}\, z$$

$$(z_1, z_2, , \ldots, z_N) = (-N+1, -N+3, \ldots, N-1)$$

$$P(0) = 1, \qquad P(1) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

$$P(2) = \frac{1}{\sqrt{4}} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix}$$

$$P(3) = \frac{1}{\sqrt{8}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & -1 & -3 \\ 3 & -1 & -1 & 3 \\ 1 & -1 & 1 & -1 \end{pmatrix},$$

Hypothesis confirmed: matrices $P = P(N-1)$ are idempotent, $P^2 = I$

$$P(4) = \frac{1}{\sqrt{16}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & 2 & 0 & -2 & -4 \\ 6 & 0 & -2 & 0 & 6 \\ 4 & -2 & 0 & 2 & -4 \\ 1 & -1 & 1 & -1 & 1 \end{pmatrix}$$

Zero order Schrödinger equation = matrix identity

$$H^{(0)}\psi^{(0)} = \psi^{(0)}\varepsilon^{(0)}$$
$$P\varepsilon^{(0)}PP = P\varepsilon^{(0)}$$

The first order problem = the left-hand-side defined

$$\varepsilon^{(1)} + P\,\Psi^{(1)}\varepsilon^{(0)} - \varepsilon^{(0)}P\,\Psi^{(1)} = P\,H^{(1)}\,P.$$

The second order problem = the upper row defined

$$\varepsilon^{(2)} + P\,\Psi^{(2)}\varepsilon^{(0)} - \varepsilon^{(0)}P\,\Psi^{(2)} =$$
$$P\,H^{(2)}\,P + P\,H^{(1)}\,\Psi^{(1)} - P\,\Psi^{(1)}\varepsilon^{(1)}$$

$N=2$ **illustration**

$$\varepsilon_{11}^{(1)} = \varepsilon_{22}^{(1)} = \beta/\sqrt{2\gamma}.$$

$$\Psi_{11}^{(1)} - \Psi_{21}^{(1)} = -\beta/\sqrt{2\gamma}$$
$$\Psi_{12}^{(1)} + \Psi_{22}^{(1)} = \beta/\sqrt{2\gamma}$$

Perturbation corrections in integer arithmetics

$$E_{1,2} = \frac{\beta}{\lambda^2} \pm \frac{2\sqrt{2\gamma}}{\lambda} + 2\beta \pm \frac{\beta^2}{\sqrt{2\gamma}}\,\lambda +$$

$$+0\cdot\lambda^2 \mp \frac{\beta^4}{8\gamma\sqrt{2\gamma}}\,\lambda^3 + 0\cdot\lambda^4 + \mathcal{O}(\lambda^5).$$

Remark: Wave functions also available from QES recurrences

Quartic anharmonic oscillator

$$q = 2: \quad V_{\{2\}}(r) = A\, r^4 + B\, r^3 + C\, r^2 + D\, r + F\, r^{-1} + G\, r^{-2}$$

Unperturbed solutions

$$Q^{(0)}(s,t) \begin{pmatrix} u_0^{(0)} \\ u_1^{(0)} \\ \vdots \\ u_N^{(0)} \end{pmatrix} = 0.$$

$$Q^{(0)}(s,t) = Q^{(0)}(0,0) + s\,\mathcal{J} + t\,\mathcal{K}$$

Two coupled secular determinants

$$\det\left[\mathcal{J}^T\, Q^{(0)}(0,t) + s\, I\right] = 0$$

$$\det\left[\mathcal{K}^T\, Q^{(0)}(s,0) + t\, I\right] = 0$$

Real roots ($n = 0, 1, \ldots, [N/2]$)

$$s = t = s^{(0)}_{[n+1]}(N) = t^{(0)}_{[n+1]}(N) = N - 3n$$

Pascal-like triangles

$$u^{(0)}_{[1],n}(N) = (-1)^{N+n} \begin{pmatrix} N \\ n \end{pmatrix}, \qquad s = s_{[1]}(N) = N$$

$$u^{(0)}_{[K+1],k}(2K) =$$

$$u^{(0)}_{[K],k}(2K-2) + u^{(0)}_{[K],k-1}(2K-2) + u^{(0)}_{[K],k-2}(2K-2).$$

Wave functions

$$\psi^{(0)}_{[1]}(r) = 1,\ \psi^{(0)}_{[2]}(r) = 1 + r/\mu + r^2/\mu^2$$

$$\psi^{(0)}_{[3]}(r) = (1 + r/\mu + r^2/\mu^2)^2$$

$$\psi^{(0)}_{[n+1]}(r) = (1-r/\mu)^{N-2n}\,(1+r/\mu+r^2/\mu^2)^n, \qquad n = 0, 1, \ldots, \left[\frac{N}{2}\right].$$

Corrections

$$\lambda = 1/(\mu\tau) = 1/\Omega$$

$$Q^{(1)} = \begin{pmatrix} 0 & 0 & & & & \\ 0 & -\gamma\mu & 1 & & & \\ & -\beta\mu^2 & -2\gamma\mu & 3 & & \\ & & -2\beta\mu^2 & -3\gamma\mu & \ddots & \\ & & & -3\beta\mu^2 & \ddots & \frac{N(N-1)}{2} \\ & & & & \ddots & -N\gamma\mu \\ & & & & & -N\beta\mu^2 \end{pmatrix}$$

$$s = s^{(0)} + \lambda\, s^{(1)} + \lambda^2 s^{(2)} + \dots$$
$$t = t^{(0)} + \lambda\, t^{(1)} + \lambda^2 t^{(2)} + \dots$$

$$\begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_N \end{pmatrix} = \begin{pmatrix} u_0^{(0)} \\ \vdots \\ u_N^{(0)} \end{pmatrix} + \lambda \begin{pmatrix} u_0^{(1)} \\ \vdots \\ u_N^{(1)} \end{pmatrix} + \lambda^2 \begin{pmatrix} u_0^{(2)} \\ \vdots \\ u_N^{(2)} \end{pmatrix} + \dots,$$

The perturbed Schrödinger equation

$$\Big[Q^{(0)}(0,0) + s^{(0)}\,\mathcal{J} + t^{(0)}\,\mathcal{K}\Big]\,\vec{u}^{(k)} +$$
$$+\Big[Q^{(1)}(0,0) + s^{(1)}\,\mathcal{J} + t^{(1)}\,\mathcal{K}\Big]\,\vec{u}^{(k-1)} +$$
$$+\ldots+\Big(s^{(k)}\,\mathcal{J} + t^{(k)}\,\mathcal{K}\Big)\,\vec{u}^{(0)} = 0.$$

Pairs of independent left eigenvectors $[\vec{v}^{[s]}]^T$ **and** $[\vec{v}^{[a]}]^T$

($\langle w_\star| \equiv \vec{\sigma}^T \mathcal{J}^T$ **and** $\langle w^\star| \equiv \vec{\theta}^T \mathcal{K}^T$ **not recommended)**

Known overlaps $Z^{(k-1)}_{[s/a]} = \langle v^{[s/a]}|\Xi^{(k-1)}\rangle$ **with kets** $|\Xi^{(k-1)}\rangle =$

$$-\Big[Q^{(1)}(0,0) + s^{(1)}\,\mathcal{J} + t^{(1)}\,\mathcal{K}\Big]\,\vec{u}^{(k-1)} -$$
$$-\ldots-\Big(s^{(k-1)}\,\mathcal{J} + t^{(k-1)}\,\mathcal{K}\Big)\,\vec{u}^{(1)}$$

Two by two equation to solve in each order

$$\langle v^{[s]}|(\mathcal{J}\vec{u}^{(0)})\rangle\, s^{(k)} + \langle v^{[s]}|(\mathcal{K}\vec{u}^{(0)})\rangle\, t^{(k)} = Z^{(k-1)}_{[s]},$$
$$\langle v^{[a]}|(\mathcal{J}\vec{u}^{(0)})\rangle\, s^{(k)} + \langle v^{[a]}|(\mathcal{K}\vec{u}^{(0)})\rangle\, t^{(k)} = Z^{(k-1)}_{[a]}.$$

Summary

The coupling and energy $\left(2s^{(k)}, 2t^{(k)}\right)$ from 2 by 2 inversion and normalization

$$\begin{pmatrix} 1/\langle v^{[s]}|(\mathcal{J}\vec{u}^{(0)})\rangle, & 1/\langle v^{[a]}|(\mathcal{J}\vec{u}^{(0)})\rangle \\ 1/\langle v^{[s]}|(\mathcal{K}\vec{u}^{(0)})\rangle, & 1/\langle v^{[a]}|(\mathcal{K}\vec{u}^{(0)})\rangle \end{pmatrix} \begin{pmatrix} Z_{[s]}^{(k-1)} \\ Z_{[a]}^{(k-1)} \end{pmatrix}$$

$$\langle v^{[s]}|(\mathcal{J}\vec{u}^{(0)})\rangle = \langle v^{[s]}|(\mathcal{K}\vec{u}^{(0)})\rangle$$
$$\langle v^{[a]}|(\mathcal{J}\vec{u}^{(0)})\rangle = -\langle v^{[a]}|(\mathcal{K}\vec{u}^{(0)})\rangle$$

One uses the renormalization ambiguity fixing

$$\vec{u}^{(k)} \to \vec{u}^{(k)} + const \times \vec{u}^{(0)},$$
$$u_0^{\star(k)} = 0 \qquad \textbf{or} \qquad \left(u_\star^{(k)}\right)_N = 0$$

and order=by-order recurrences with known kets

$$|\tau^{(k-1)}\rangle = |\Xi^{(k-1)}\rangle - s^{(k)}\,\mathcal{J}\vec{u}^{(0)} - t^{(k)}\,\mathcal{K}\vec{u}^{(0)}$$

Wave functions from having dropped the last two lines

$$R^{\star}\begin{pmatrix} u_1^{\star(k)} \\ u_2^{\star(k)} \\ \vdots \\ u_N^{\star(k)} \end{pmatrix} = \begin{pmatrix} \tau_0^{\star(k-1)} \\ \tau_1^{\star(k-1)} \\ \vdots \\ \tau_{N-1}^{\star(k-1)} \end{pmatrix}$$

$$R^{\star} = \begin{pmatrix} 1 & & & & & \\ s^{(0)} & 2 & & & & \\ t^{(0)} & s^{(0)} & 3 & & & \\ N-1 & t^{(0)} & s^{(0)} & 4 & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & 3 & t^{(0)} & s^{(0)} & N \end{pmatrix}$$

i.e., triangular unperturbed propagators

$$\begin{pmatrix} u_{\star 1}^{(k)} \\ \vdots \\ u_{\star N}^{(k)} \end{pmatrix} = (R_{\star})^{-1} \begin{pmatrix} \tau_{\star 0}^{(k-1)} \\ \vdots \\ \tau_{\star N-1}^{(k-1)} \end{pmatrix}.$$

Table 1: All roots $s^{(0)} = t^{(0)} = A/2 + i\sqrt{3}B/2$

N	1		2				3					
A	2	-1	4	1	-2	-2	6	3	0	0	-3	-3
B	0	$\pm$ 1	0	$\pm$ 1	0	$\pm$ 2	0	$\pm$ 1	0	$\pm$ 2	$\pm$ 1	$\pm$ 3
N	4											
A	8	5	2	2	-1	-1	-4	-4	-4			
B	0	$\pm$ 1	0	$\pm$ 2	$\pm$ 1	$\pm$ 3	0	$\pm$ 2	$\pm$ 4			
N	5											
A	10	7	4	4	1	1	-2	-2	-2	-5	-5	-5
B	0	$\pm$ 1	0	$\pm$ 2	$\pm$ 1	$\pm$ 3	0	$\pm$ 2	$\pm$ 4	$\pm$ 1	$\pm$ 3	$\pm$ 5

Table 2: Real eigenvectors of our pseudo-Hamiltonians $Q^{(0)}$.

N	$s^{(0)}$	$u_0^{(0)}$	$u_1^{(0)}$	$u_2^{(0)}$	$u_3^{(0)}$	$u_4^{(0)}$	$u_5^{(0)}$	$u_6^{(0)}$
0	0	1	-	-	-	-	-	-
1	1	1	-1	-	-	-	-	-
2	2	1	-2	1	-	-	-	-
	-1	1	1	1	-	-	-	-
3	3	1	-3	3	-1	-	-	-
	0	1	0	0	-1	-	-	-
4	4	1	-4	6	-4	1	-	-
	1	1	-1	0	-1	1	-	-
	-2	1	2	3	2	1	-	-
5	5	1	-5	10	-10	5	-1	-
	2	1	-2	1	-1	2	-1	-
	-1	1	1	1	-1	-1	-1	-
6	6	1	-6	15	-20	15	-6	1
	3	1	-3	3	-2	3	-3	1
	0	1	0	0	-2	0	0	1
	-3	1	3	6	7	6	3	1

Table 3: Generalized Pascal triangle (ground states: $N = 2K$, $s_{[K+1]} = -K$).

K	...	$u_{K-4}^{(0)}$	$u_{K-3}^{(0)}$	$u_{K-2}^{(0)}$	$u_{K-1}^{(0)}$	$u_K^{(0)}$	$u_{K+1}^{(0)}$	$u_{K+2}^{(0)}$	$u_{K+3}^{(0)}$	$u_{K+4}^{(0)}$	...
0	...	0	0	0	0	1	0	0	0	0	...
1	...	0	0	0	1	1	1	0	0	0	...
2	...	0	0	1	2	3	2	1	0	0	...
3	...	0	1	3	6	7	6	3	1	0	...
4	...	1	4	10	16	19	16	10	4	1	...
5	...	5	15	30	45	51	45	30	15	5	...
6	...	21	50	90	126	141	126	90	50	21	...
...						...					

Table 4: Pairs of independent left eigenvectors of $Q^{(0)}(s,s)$.

		sym. and antisym. v_k						shortened σ_k and θ_k					overlaps
	$k=$	0	1	2	3	4	5	0	1	2	3	4	
N	s												$\langle v\|(\mathcal{J}\vec{u}^{(0)})\rangle$
0	0	1	1	-	-	-	-	1	-	-	-	-	1
		1	-1	-	-	-	-	1	-	-	-	-	1
1	1	1	-2	1	-	-	-	1	-1	-	-	-	3
		1	0	-1	-	-	-	- 1	1	-	-	-	1
2	2	2	-1	-1	2	-	-	0	1	-1	-	-	3
		0	1	-1	0	-	-	-	-	-	-	-	-3
	-1	1	1	1	1	-	-	3	1	2	-	-	3
		3	-1	1	-3	-	-	2	1	3	-	-	3
3	3	1	1	-2	1	1	-	1	0	-1	1	-	-9
		1	-1	0	1	-1	-	1	-1	0	1	-	3
	0	2	-1	0	-1	2	-	2	0	0	-1	-	3
		2	1	0	-1	-2	-	- 1	0	0	2	-	3
4	4	1	-2	1	1	-2	1	1	-1	0	1	-1	9
		1	0	-1	1	0	-1	- 1	1	0	-1	1	-9
	1	7	1	-2	-2	1	7	7	-3	-1	-1	4	9
		1	-1	0	0	1	-1	4	-1	-1	-3	7	3
	-2	1	1	1	1	1	1	3	1	2	1	2	9
		3	-1	1	-1	1	-3	2	1	2	1	3	3