$\mathcal{PT}-$supersymmetric partner of a short-range square well

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In a box of size L, a spatially antisymmetric square-well potential of a purely imaginary strength ig and size $\ell < L$ is interpreted as an initial element of the SUSY hierarchy of solvable Hamiltonians, the energies of which are all real at $\ell < \ell_{\max}(g) \leq L$. The first partner potential is constructed in closed form and discussed.

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1 Introduction

The technically slightly complicated but quantum-mechanically straightforward solution of the one-dimensional and $\mathcal{PT}-$symmetric Schrödinger equation

$$\left[-\frac{d^2}{dx^2} + V(x) - D_0\right]\psi_n(x) = E_n\psi_n(x), \qquad n = 1, 2, \ldots \tag{1}$$

with the Dirichlet boundary conditions $\psi(\pm L) = 0$ and with the purely imaginary $V(x)$ may be found elsewhere [1, 2, 3]. Here, such a model with real spectrum and

$$V(x) = V^{(+)}(x) = \left\{ \begin{array}{ll} 0 & \\ \mathrm{i}\, g \,\mathrm{sign}\, x, & g > 0 \end{array} \right. \qquad \text{for} \qquad \left\{ \begin{array}{l} L > |x| > l \\ |x| \leq l, \end{array} \right.$$

will be considered factorized and complemented by another, similar model,

$$-\frac{d^2}{dx^2} + V^{(+)}(x) - D_0 = \bar{\mathcal{A}}\mathcal{A} \equiv H^{(+)}, \qquad \mathcal{A}\bar{\mathcal{A}} = -\frac{d^2}{dx^2} + V^{(-)}(x) - D_0 \equiv H^{(-)}$$

where the well known operators and identities are employed,

$$\mathcal{A} = \frac{d}{dx} + W(x), \qquad \bar{\mathcal{A}} = -\frac{d}{dx} + W(x), \qquad V^{(\pm)} - D_0 = W^2 \mp W' . \tag{2}$$

By symbols $\psi_n^{(+)}(x)$ (resp. $\psi_n^{(-)}(x)$), $n = 0, 1, 2, \ldots$ we denote the wave functions of $H^{(+)}$ (resp. $H^{(-)}$) and have the so called unbroken-supersymmetry condition $\mathcal{A}\,\psi_0^{(+)}(x) = 0$ of the Witten's superymmetric quantum mechanics (SUSY QM, [4]). As long as the latter formalism usually does not work with non-Hermitian operators, we believe that both the construction and some unusual properties of the partner potential $V^{(-)}(x)$ deserve an explicit description.

2 The $\mathcal{PT}-$symmetric SUSY partner potential $V^{(-)}(x)$

The purpose of the present section is to construct and study the SUSY partner $H^{(-)}$ of the square-well Hamiltonian $H^{(+)}$ in the physically-relevant unbroken $\mathcal{PT}$-symmetry regime, corresponding to $g < g_c(l)$ of ref. [1].

2.1 The determination of the parameters

Let us denote the four regions $-L < x < -l$, $-l < x < 0$, $0 < x < l$, $l < x < L$ by $L2$, $L1$, $R1$, $R2$, respectively. Identifying $V^{(+)}$ with our square-well potential above, i.e., $V_{L2}^{(+)}(x) = 0$, $V_{L1}^{(+)}(x) = -\mathrm{i}g$, $V_{R1}^{(+)}(x) = \mathrm{i}g$, $V_{R2}^{(+)}(x) = 0$ we may set $D_0 = k_0^2 = t_0^2 - s_0^2 = -\kappa_0^2 + \mathrm{i}g$ and obtain for the superpotential and the partner potential the respective formulae

$$W(x) = \begin{cases} W_{L2}(x) = k_0 \tan[k_0(x + x_{L2})] \\ W_{L1}(x) = -\kappa_0^* \tanh[\kappa_0^*(x + x_{L1})] \\ W_{R1}(x) = -\kappa_0 \tanh[\kappa_0(x - x_{R1})] \\ W_{R2}(x) = k_0 \tan[k_0(x - x_{R2})] \end{cases} \tag{1}$$

and

$$V^{(-)}(x) = \begin{cases} V_{L2}^{(-)}(x) = 2k_0^2 \sec^2[k_0(x + x_{L2})] \\ V_{L1}^{(-)}(x) = -2\kappa_0^{*2} \operatorname{sech}^2[\kappa_0^*(x + x_{L1})] - \mathrm{i}g \\ V_{R1}^{(-)}(x) = -2\kappa_0^2 \operatorname{sech}^2[\kappa_0(x - x_{R1})] + \mathrm{i}g \\ V_{R2}^{(-)}(x) = 2k_0^2 \sec^2[k_0(x - x_{R2})] \end{cases} . \tag{2}$$

Here x_{L2}, x_{L1}, x_{R1} and x_{R2} denote four integration constants. We choose

$$x_{L2} = L + \frac{\pi}{2k_0}, \qquad x_{R2} = L - \frac{\pi}{2k_0} \tag{3}$$

to ensure that $V_{L2}^{(-)}$ and $V_{R2}^{(-)}$ blow up at the end points $x = -L$ and $x = L$. This is in tune with [5]. We thus get

$$V_{L2}^{(-)}(x) = 2k_0^2 \csc^2[k_0(x + L)] \qquad V_{R2}^{(-)}(x) = 2k_0^2 \csc^2[k_0(x - L)]. \tag{4}$$

Observe that for the superpotential, $W_{L2}(x)$ and $W_{R2}(x)$ also blow up at these points:

$$W_{L2}(x) = -k_0 \cot[k_0(x + L)] \qquad W_{R2}(x) = -k_0 \cot[k_0(x - L)]. \tag{5}$$

Due to the unbroken-SUSY condition the ground-state wavefunction of $H^{(+)}$ is

$$\psi^{(+)}_{0R2}(x) = \psi^{(+)*}_{0L2}(-x) = A^{(+)}_0 \sin[k_0(L-x)], \tag{6}$$
$$\psi^{(+)}_{0R1}(x) = \psi^{(+)*}_{0L1}(-x) = B^{(+)}_0 \cosh(\kappa_0 x) + \mathrm{i}\frac{C^{(+)}_0}{\kappa_0 l}\sinh(\kappa_0 x). \tag{7}$$

The unbroken-supersymmetry condition is satisfied due to the choice made for the integration constants x_{R2}, x_{L2} in equation (3). In the region $R1$ we find a condition fixing the value of x_{R1},

$$\tanh(\kappa_0 x_{R1}) = -\frac{\mathrm{i}C^{(+)}_0}{\kappa_0 l B^{(+)}_0} = \frac{k_0\cot[k_0(L-l)]\coth(\kappa_0 l)+\kappa_0}{k_0\cot[k_0(L-l)]+\kappa_0\coth(\kappa_0 l)} \tag{8}$$

where in the last step we used ref. [1]. A similar relation applies in $L1$, thus leading to the result

$$x_{L1} = x^*_{R1}. \tag{9}$$

Note that in contrast with the real integration constants x_{R2}, x_{L2}, the constants x_{R1} and x_{L1} are complex. Separating both sides of equation (8) into a real and an imaginary part, we obtain the two equations

$$\frac{\sinh X\cosh X}{\cosh^2 X\cos^2 Y+\sinh^2 X\sin^2 Y} = \frac{N^r}{D} \tag{10}$$
$$\frac{\sin Y\cos Y}{\cosh^2 X\cos^2 Y+\sinh^2 X\sin^2 Y} = \frac{N^i}{D} \tag{11}$$

where we have used the decompositions $\kappa_0 = s_0 + \mathrm{i}t_0$, $x_{R1} = x^r_{R1} + \mathrm{i}x^i_{R1}$, $\kappa_0 x_{R1} = X + \mathrm{i}Y$, implying that

$$X = s_0 x^r_{R1} - t_0 x^i_{R1} \qquad Y = t_0 x^r_{R1} + s_0 x^i_{R1} \tag{12}$$

and we have defined

$$N^r = \{-s_0^2\cos[2k_0(L-l)]+t_0^2\}\sinh(2s_0 l) + k_0 s_0\sin[2k_0(L-l)]\cosh(2s_0 l) \tag{13}$$

$$N^i = \{s_0^2 - t_0^2\cos[2k_0(L-l)]\}\sin(2t_0 l) - k_0 t_0\sin[2k_0(L-l)]\cos(2t_0 l) \tag{14}$$

$$\begin{aligned} D \;=\; & \{-s_0^2\cos[2k_0(L-l)]+t_0^2\}\cosh(2s_0 l) + \{s_0^2 - t_0^2\cos[2k_0(L-l)]\}\cos(2t_0 l) \\ & + k_0\sin[2k_0(L-l)][s_0\sinh(2s_0 l)+t_0\sin(2t_0 l)]. \end{aligned} \tag{15}$$

Equations (10) and (11), when solved numerically, furnish the values of both the parameters x^r_{R1} and x^i_{R1}. One may also observe that the resulting superpotential $W(-x) = -W^*(x)$ and partner potential $V^{(-)}(-x) = V^{(-)*}(x)$ are $\mathcal{PT}$-antisymmetric and $\mathcal{PT}$-symmetric, respectively.

2.2 Eigenfunctions in the partner potential

On exploiting the SUSY intertwining relations, the eigenfunctions $\psi_n^{(-)}(x)$, $n = 0$, 1, 2, ..., of $H^{(-)}$ can be obtained by acting with $\mathcal{A}$ on $\psi_{n+1}^{(+)}(x)$, subject to the preservation of the boundary and continuity conditions

$$
\begin{aligned}
\psi_{nL2}^{(-)}(-L) &= 0 \qquad \psi_{nR2}^{(-)}(L) = 0 && (16)\\
\psi_{nL2}^{(-)}(-l) &= \psi_{nL1}^{(-)}(-l) \qquad \partial_x\psi_{nL2}^{(-)}(-l) = \partial_x\psi_{nL1}^{(-)}(-l) && (17)\\
\psi_{nL1}^{(-)}(0) &= \psi_{nR1}^{(-)}(0) \qquad \partial_x\psi_{nL1}^{(-)}(0) = \partial_x\psi_{nR1}^{(-)}(0) && (18)\\
\psi_{nR1}^{(-)}(l) &= \psi_{nR2}^{(-)}(l) \qquad \partial_x\psi_{nR1}^{(-)}(l) = \partial_x\psi_{nR2}^{(-)}(l). && (19)
\end{aligned}
$$

Application of $\mathcal{A}$ leads to the forms

$$
\begin{aligned}
\psi_{nL2}^{(-)}(x) &= C_{nL2}^{(-)}\, A_{n+1}^{(+)*} \sin[k_{n+1}(L+x)] \\
&\quad \times \{k_{n+1}\cot[k_{n+1}(L+x)] - k_0\cot[k_0(L+x)]\} && (20)\\
\psi_{nL1}^{(-)}(x) &= C_{nL1}^{(-)}\, B_{n+1}^{(+)} \sinh(\kappa_{n+1}^* x)\{\kappa_{n+1}^* - \kappa_0^*\tanh[\kappa_0^*(x+x_{R1}^*)]\coth(\kappa_{n+1}^* x)\} \\
&\quad + C_{nL1}^{(-)} \frac{\mathrm{i}C_{n+1}^{(+)}}{\kappa_{n+1}^* l}\sinh(\kappa_{n+1}^* x) \\
&\quad \times \{\kappa_{n+1}^*\coth(\kappa_{n+1}^* x) - \kappa_0^*\tanh[\kappa_0^*(x+x_{R1}^*)]\} && (21)\\
\psi_{nR1}^{(-)}(x) &= C_{nR1}^{(-)}\, B_{n+1}^{(+)} \sinh(\kappa_{n+1} x)\{\kappa_{n+1} - \kappa_0\tanh[\kappa_0(x-x_{R1})]\coth(\kappa_{n+1} x)\} \\
&\quad + C_{nR1}^{(-)} \frac{\mathrm{i}C_{n+1}^{(+)}}{\kappa_{n+1} l}\sinh(\kappa_{n+1} x) \\
&\quad \times \{\kappa_{n+1}\coth(\kappa_{n+1} x) - \kappa_0\tanh[\kappa_0(x-x_{R1})]\} && (22)\\
\psi_{nR2}^{(-)}(x) &= C_{nR2}^{(-)}\, A_{n+1}^{(+)} \sin[k_{n+1}(L-x)] \\
&\quad \times \{-k_{n+1}\cot[k_{n+1}(L-x)] + k_0\cot[k_0(L-x)]\} && (23)
\end{aligned}
$$

where $C_{nL2}^{(-)}$, $C_{nL1}^{(-)}$, $C_{nR1}^{(-)}$, $C_{nR2}^{(-)}$ denote some complex constants and equation (9) has been used. Boundary conditions (16) are satisfied. It remains to impose the continuity conditions (17) – (19).

Let us first match the regions $L1$ and $R1$ at $x = 0$. Since equation (8) implies two constraints

$$
\begin{aligned}
\kappa_0\tanh(\kappa_0 x_{R1}) &= -\kappa_0^*\tanh(\kappa_0^* x_{R1}^*), && (24)\\
\kappa_{n+1}^{*2} - \kappa_{n+1}^2 &= \kappa_0^{*2} - \kappa_0^2 = -2\mathrm{i}\, g, && (25)
\end{aligned}
$$

the continuity conditions (18) yield the two relations which are mutually compatible and lead to the condition

$$C_{nR1}^{(-)} = C_{nL1}^{(-)}. \qquad (26)$$

Considering next the matching between $R1$ and $R2$ at $x = l$, we obtain from equation (19) the two conditions

$$C_{nR1}^{(-)}\{k_{n+1}\cot[k_{n+1}(L-l)] + \kappa_0\tanh[\kappa_0(l-x_{R1})]\}$$

$$= C^{(-)}_{nR2}\{k_{n+1}\cot[k_{n+1}(L-l)] - k_0\cot[k_0(L-l)]\} \tag{27}$$

$$\begin{aligned} C^{(-)}_{nR1}\Big(&\kappa^2_{n+1} - \kappa^2_0 + \kappa_0\tanh[\kappa_0(l-x_{R1})]\{k_{n+1}\cot[k_{n+1}(L-l)] \\ &+ \kappa_0\tanh[\kappa_0(l-x_{R1})]\}\Big) \\ = C^{(-)}_{nR2}\Big(&k^2_0 - k^2_{n+1} - k_0\cot[k_0(L-l)]\{k_{n+1}\cot[k_{n+1}(L-l)] \\ &- k_0\cot[k_0(L-l)]\}\Big) \end{aligned} \tag{28}$$

after making use of equations of ref. [1] to eliminate $A^{(+)}_{n+1}$, $B^{(+)}_{n+1}$ and $C^{(+)}_{n+1}$. Equations (27) and (28) both yield the same result

$$C^{(-)}_{nR1} = C^{(-)}_{nR2} \tag{29}$$

due to the two relations

$$\kappa_0\tanh[\kappa_0(l-x_{R1})] = -k_0\cot[k_0(L-l)] \tag{30}$$

and

$$\kappa^2_{n+1} - \kappa^2_0 = k^2_0 - k^2_{n+1} \tag{31}$$

deriving from (8).

Since a result similar to (29) applies at the interface between regions $L2$ and $L1$, we conclude that the partner potential eigenfunctions are given by equations (20) – (23) with

$$C^{(-)}_{nL2} = C^{(-)}_{nL1} = C^{(-)}_{nR1} = C^{(-)}_{nR2} \equiv C^{(-)}_n. \tag{32}$$

Such eigenfunctions are $\mathcal{PT}$-symmetric provided we choose $C^{(-)}_n$ imaginary:

$$C^{(-)*}_n = -C^{(-)}_n. \tag{33}$$

3 Discontinuities in the partner potential $V^{(-)}(x)$

In subsection 2.1, we have constructed the SUSY partner $V^{(-)}(x)$ of a piecewise potential with three discontinuities at $x = -l$, 0 and l. We may now ask the following question: does the former have the same discontinuities as the latter or could the discontinuity number decrease? We plan to prove here that the second alternative can be ruled out.

For such a purpose, we will examine successively under which conditions $V^{(-)}(x)$ could be continuous at $x = l$ or at $x = 0$ and we will show that such restrictions would not be compatible with some relations deriving from our unbroken-SUSY assumption. Observe that we do not have to study continuity at $x = -l$ separately, since $V^{(-)}(x)$ being $\mathcal{PT}$-symmetric must be simultaneously continuous or discontinuous at $x = -l$ and $x = l$.

Let us start with the point $x = l$. Matching there $V_{R1}^{(-)}(x)$ and $V_{R2}^{(-)}(x)$, given in equations (2) and (4), respectively, leads to the relation

$$-2\kappa_0^2 \operatorname{sech}^2[\kappa_0(l - x_{R1})] + \mathrm{i}g = 2k_0^2 \csc^2[k_0(L - l)]. \tag{34}$$

On using (30) and some simple trigonometric identities, such a relation can be transformed into $k_0^2 = -\kappa_0^2 + \frac{1}{2}\mathrm{i}g$, which manifestly contradicts our definition of κ. Hence continuity of $V^{(-)}(x)$ at $x = l$ is ruled out.

Consider next the point $x = 0$. On equating $V_{R1}^{(-)}(0)$ with $V_{L1}^{(-)}(0)$ and employing (2) and (9), we obtain the condition

$$-2\kappa_0^2 \operatorname{sech}^2(\kappa_0 x_{R1}) + \mathrm{i}g = -2\kappa_0^{*2} \operatorname{sech}^2(\kappa_0^* x_{R1}^*) - \mathrm{i}g. \tag{35}$$

Equation (8) then yields the relation $-\kappa_0^2 + \frac{1}{2}\mathrm{i}g = -\kappa_0^{*2} - \frac{1}{2}\mathrm{i}g$, which contradicts the definition of κ again. Continuity of $V^{(-)}(x)$ at $x = 0$ is therefore excluded too.

We conclude that under the simplest assumption of unbroken SUSY with a factorization energy equal to the ground-state energy of $H^{(+)}$, the partner potential $V^{(-)}(x)$ has the same three discontinuities at $x = -l$, 0 and l as $V^{(+)}(x)$.

We also observe that the SUSY partners $H^{(\pm)}$ may remain both non-Hermitian and $\mathcal{PT}$-symmetric. Charge operator $\mathcal{C}$ [6] may be constructed in the specific form which differs from the unit operator mostly in a finite-dimensional subspace of the Hilbert space [7]. This is one of the most important merits of all the square-well models with $L < \infty$. It seems to open a new inspiration for a direct physical applicability of non-Hermitian models whenever their spectrum remains real.

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