Seminář v Bratislavě (Tuesday, May 15, 2007, 14.00)

Modelování nestabilit

v kvazi-hermitovské kvantovém mechanice

neboli taky, v překladu do angličtiny,

What should we all know about

pseudo-Hermitian models in quantum mechanics

Miloslav Znojil, Nuclear Physics Institute, 250 68 Řež, Czech Republic

email: znojil@ujf.cas.cz

A.

INSTABILITIES

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A. I.

A small sample of quantum instabilities

- strongly attractive Coulomb (relativistic)
- too attractive centrifugal force (nonrelativistic)
- 2D oscillator in cranking regime [$\omega_x = 3$, $\omega_x = 2$, $\Omega \in (2, 3)$]

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A. II.

The respective reasons of these instabilities

- (Dirac Coulomb): **antiparticles**, i.e., **physics** enters the scene
- (centrifugal $\ell \to -\frac{1}{2}$): $\mathcal{PT}-$**symmetry** enters the scene (Znojil 1999)
- (cranking): **pseudo-Hermiticity** enters the scene

  (W. D. Heiss and R. Nazmitdinov, 2007)

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A. III.

What is $\mathcal{PT}-$symmetry ?

- (Buslaev and Grecchi 1993): symmetry of $H = p^2 + \omega^2 x^2 - x^4$
- (Bender and Boettcher 1998): symmetry of $H = p^2 - (\mathrm{i}x)^{2+\varepsilon}$ with $\varepsilon \geq 0$
- (Mostafazadeh 2002): special case of $\mathcal{P}-$**pseudo-Hermiticity**
- (Znojil 2005): special case of **quantum toboggans**

  (Phys. Lett. A 342 (2005) 36 - 47 and

  J. Phys. A: Math. Gen. 39 (2006) 13325 - 13336)

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A. IV.

What is pseudo-Hermiticity ?

- (textbooks): symmetry of H such that $H^\dagger = \mathcal{P}\, H\, \mathcal{P}^{-1}$, with $\mathcal{P} = \mathcal{P}^\dagger$
- (Mostafazadeh 2002): $\mathcal{P}$ need not be known
- (Solombrino 2002/ Znojil 2006): $\mathcal{P}$ need not be self-adjoint

  ($\Longrightarrow$ the "weak"/"strengthened" pseudo-Hermiticity)

  (M.Z., Phys. Lett. A 353 (2006) 463 - 468 and
  J. Phys. A: Math. Gen. 39 (2006) 4047 - 4061)

B.

MODELS

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B. I.

Are we still *inside* quantum mechanics?

- answer (Scholtz, Geyer and Hahne 1992): YES.

  (F.G.S., H.B.G. and F.J.W.H., Ann. Phys. 213 (1992) 74)
- re-answered: Mostafazadeh 2002, Bender et al 2002, Znojil 2004
- essence: there exists **another** symmetry of H such that $H^\dagger = \Theta\, H\, \Theta^{-1}$, $\Theta > 0$
- varying notation: η_+, $\mathcal{CP}$, $\exp Q$, $\mathcal{PQ}$. The series of dedicated conferences:

  ($\Longrightarrow$ http://gemma.ujf.cas.cz/˜ znojil)
- the most recent review: C. Bender, hep-th/0703096.

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B. II.

ODEs with $x \in (-\infty, \infty)$:

$$H^{(BB)}(\nu) = -\frac{d^2}{dx^2} + g(x)\,x^2\,, \qquad g(x) = \omega^2 + (\mathrm{i}x)^\nu\,, \qquad \nu \geq 0$$

typically, $\mathcal{PT}-$*symmetric:*

$$H = -\frac{d^2}{dx^2} + U(x) + \mathrm{i}\,W(x) \neq H^\dagger, \quad U(x) = U(-x)\,, \quad W(x) = -W(-x)\,,$$

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B. III.

in the basis:

$$| \psi_+ \rangle = \sum_{m=0}^{N_+} | 2m \rangle \, \phi_m, \quad | \psi_- \rangle = \sum_{m=0}^{N_-} | 2m+1 \rangle \, \chi_m$$

i.e., parity-partitioned, complex, symmetric matrices:

$$\tilde{H} = \begin{pmatrix} S & \mathrm{i}\, B \\ \mathrm{i}\, B^T & L \end{pmatrix},$$

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B. IV.

normalization trick $|\,\psi\rangle = |\,\psi_+\rangle - \mathrm{i}\,|\,\psi_-\rangle$

$\heartsuit$: *equivalence to the asymmetric* **real** *matrices:*

$$H = \begin{pmatrix} S & +B \\ -B^T & L \end{pmatrix}, \qquad |n\rangle = \begin{pmatrix} \vec{\phi}_n \\ \vec{\chi}_n \end{pmatrix}, \qquad |n\rangle\!\rangle = \begin{pmatrix} \vec{\phi}_n \\ -\vec{\chi}_n \end{pmatrix}.$$

$\diamondsuit$: *reality of spectra, quasihermiticity:*

$$H^\dagger = \Theta\, H\, \Theta^{-1}, \qquad I \neq \Theta = \Theta^\dagger > 0.$$

with $H = \sum_n |n\rangle \frac{E_n}{\langle\langle n|n\rangle} \langle\langle n|$ and $\Theta = \sum_n |n\rangle\rangle\, t_n\, \langle\langle n|$

using **two definitions:** $H|n\rangle = E_n|n\rangle$ **and** $\langle\langle n|H = E_n\langle\langle n|$

♠ : *exactly solvable two-by-two example*

$$\begin{pmatrix} s & b \\ -b & l \end{pmatrix} \begin{pmatrix} \phi \\ \chi \end{pmatrix} = E \begin{pmatrix} \phi \\ \chi \end{pmatrix}$$

$l - s = 2$, shift $s = -1$, $l = 1$, get $E = E_{\pm} = \sqrt{1-b^2}$, $\mathcal{D}^{(1)} = (-1, 1)$

♣ the metric:

$$\Theta = \begin{pmatrix} a & b \\ b & d \end{pmatrix} \Longrightarrow 2bT = -B(a+d)$$

positivity $\theta_{1,2} > 0 \Longleftrightarrow b \neq 0 \neq a + d = 2Z$

reparametrize $a = Z(1+\xi)$, $d = Z(1-\xi)$,

family $1 > \sqrt{\xi^2 + \sin^2\alpha}, \quad 0 \le \xi < \cos\alpha$

C.

MATRICES

C. I.

Runge Kutta square wells (not today):

$$
H^{(RK)} = \left[\begin{array}{ccccc} \mathrm{i}\,V(x_{-2}) & h & 0 & 0 & 0 \\ h & \mathrm{i}\,V(x_{-1}) & h & 0 & 0 \\ 0 & h & \mathrm{i}\,V(x_0) & h & 0 \\ 0 & 0 & h & \mathrm{i}\,V(x_1) & h \\ 0 & 0 & 0 & h & \mathrm{i}\,V(x_2) \end{array} \right] .
$$

- see the paper M. Znojil, J. Phys. A: Math. Gen. 39 (2006) 10247

  and also M. Znojil, quant-ph/0704.0214 [hep-th] (2. 4. 2007))
- Chebyshev eigenvectors at a constant potential V,

  partitioning and solution by the matching method.

C. II. Variational anharmonic (today):

HO plus a *manifestly non-selfadjoint* and *nonlocal* term,

$$H^{(AHO)} = H^{(HO)} + \sum_{\substack{m,\,n\,=\,1\\ |m\,-\,n|\,=\,1}}^{N} |\, m^{(HO)}\rangle\, h(m,n)\, \langle n^{(HO)}\,|$$

$$H^{(HO)} = -\frac{d^2}{dx^2} + x^2 = \sum_{n=1}^{\infty} |\, n^{(HO)}\rangle\, (2n-1)\, \langle n^{(HO)}\,|$$

plus "U-D symmetry",

$$H^{(N)} = \left[\begin{array}{cccccc} 1-N & g_1 & 0 & 0 & \ldots & 0 \\ -g_1 & 3-N & g_2 & 0 & \ldots & 0 \\ 0 & -g_2 & 5-N & \ddots & \ddots & \vdots \\ 0 & 0 & -g_3 & \ddots & g_2 & 0 \\ \vdots & \vdots & \ddots & \ddots & N-3 & g_1 \\ 0 & 0 & \ldots & 0 & -g_1 & N-1 \end{array} \right]$$

C.III. $N = 3$:

Benefitial effect of a new degree of freedom

♡ **idea:** regularize $H^{(2)}$ near $a^{(EP)} = \pm 1$,

$$H^{(2)} = \left(\begin{array}{cc} -1 & a \\ -a & 1 \end{array} \right) \longrightarrow H^{(3)} = \left(\begin{array}{c|cc} 3+c & 0 & b \\ \hline 0 & -1 & a \\ -b & -a & 1 \end{array} \right)$$

♢ **we know:** at $c = 0$ and $b > 0$, $\det(H^{(3)} - E) = 0$ means

$$-E^3 + 3\,E^2 + \left(-a^2 + 1 - b^2\right) E - 3 + 3\,a^2 - b^2 = 0\,.$$

and conclude: **energies real again**:

Miloslav Znojil, A return to observability near exceptional points in a schematic PT-symmetric model Phys. Lett. B 647 (2007) 225 - 230 (quant-ph/0701232).

♣ **EPs $\partial\mathcal{D}^{(3)}$, 1st step:** at $ab=0$,

$$a \in \mathcal{D}^{(3)}\Big|_{b=0} = (-1,1)\,, \qquad b \in \mathcal{D}^{(3)}\Big|_{a=0} = (-1,1).$$

the square of pairwise mergers of energies

$$(a,b) \in \{\,(1,0),\ (0,1),\ (-1,0),\ (0,-1)\,\},$$

2nd step: triple mergers of the energy levels,

$$\left\{(E-z)^3=0\right\} \implies \left\{-3 = 1-a^2-b^2, \qquad 1=-3+3\,a^2-b^2\right\}\,.$$

a bigger square of the four triple-energy-mergers,

$$(a,b) \in \{\,(\sqrt{2},\sqrt{2}),\ (-\sqrt{2},\sqrt{2}),\ (-\sqrt{2},-\sqrt{2}),\ (\sqrt{2},-\sqrt{2})\,\}.$$

3rd step, at $b>a>0$:

doubly degenerate $z=1+\beta$, $\beta \in (0,1)$ (merger of $E_{1,2}=1,3$)

and an "observer" energy $y = -1 + 2\alpha$, $\alpha > 0$

$$-(E-z)^2(E-y) = -E^3 + (2z+y)\,E^2 - \left(z^2 + 2yz\right) E + yz^2 = 0$$

gives $\alpha + \beta = 1$ plus a set of two equations,

$$a^2 + b^2 = 4 - 3\beta^2, \qquad 3a^2 - b^2 = 4 - 3\beta^2 - 2\beta^3$$

and the desired one-parametric definition of $\partial\mathcal{D}^{(3)}$:

$$a = a_\pm = \pm\sqrt{\frac{1}{2}\,(4 - 3\beta^2 - \beta^3)}, \qquad b = b_\pm = \pm\sqrt{\frac{1}{2}\,(4 - 3\beta^2 + \beta^3)}$$

with $\beta \in (-1, 1)$.

FOUR channels, discouraging

Schrödinger-equation

$$\left(\begin{array}{cc|cc} -3 & 0 & c & b \\ 0 & 1 & a & d \\ \hline -c & -a & -1 & 0 \\ -b & -d & 0 & 3 \end{array}\right)\left(\begin{array}{c} \phi_0 \\ \phi_1 \\ \hline \chi_0 \\ \chi_1 \end{array}\right) = E \left(\begin{array}{c} \phi_0 \\ \phi_1 \\ \hline \chi_0 \\ \chi_1 \end{array}\right)$$

i.e., an exactly solvable secular equation

$$E^4 - \left(10 - a^2 - b^2 - c^2 - d^2\right) E^2 - 4 \left(c^2 - d^2\right) E + C(a, b, c, s) = 0\,,$$

$$C(a,b,c,d) = 9 - 9\,a^2 - b^2 + 3\,c^2 + 3\,d^2 + a^2b^2 + c^2d^2 - 2\,abcd\,.$$

For $(E-z)^4 = E^4 = 0$ giving $c^2 = d^2$ and $C(a,b,c,d) = 0$, i.e., $\left(d^2 - \alpha\right)\left(d^2 - \beta\right) = 0$ where $\alpha = (b+3)(a-1)$ and $\beta = (b-3)(a+1)$. Thus,

$$a^2 + c^2 + b^2 + d^2 = 10\,.$$

All the domain $\mathcal{D}$ lies inside a circumscribed hypersphere. Cf.:

Miloslav Znojil, Determination of the domain of the admissible matrix elements in the four-dimensional PT-symmetric anharmonic model Phys. Lett. A, in print (available online) (quant-ph/0703168).

D.

Domains $\mathcal{D}$ of quasihermiticity

♡ four by four model has similar $\mathcal{D}^{(4)}$ since

$$\det \begin{bmatrix} 3-E & b & 0 & 0 \\ -b & 1-E & a & 0 \\ 0 & -a & -1-E & b \\ 0 & 0 & -b & -3-E \end{bmatrix} = 0$$

$$s^2 + \left(-10 + 2\,b^2 + a^2\right) s + 9 + 6\,b^2 - 9\,a^2 + b^4 = 0$$

$$s = s_{\pm} = 5 - b^2 - 1/2\,a^2 \pm 1/2\,\sqrt{64 - 64\,b^2 + 16\,a^2 + 4\,b^2 a^2 + a^4}\,.$$

$\diamondsuit$ **five by five model has similar $\mathcal{D}^{(5)}$ since**

$$H^{(5)} = \begin{bmatrix} 4 & b & 0 & 0 & 0 \\ -b & 2 & a & 0 & 0 \\ 0 & -a & 0 & a & 0 \\ 0 & 0 & -a & -2 & b \\ 0 & 0 & 0 & -b & -4 \end{bmatrix} .$$

$$-s^2 + \left(20 - 2\,b^2 - 2\,a^2\right) s - 64 - 16\,b^2 + 32\,a^2 - b^4 - 2\,a^2 b^2 = 0\,.$$

$$E_{\pm 1} = \pm\sqrt{10 - a^2 - b^2 - \sqrt{36 + 12\,a^2 + a^4 - 36\,b^2}}\,,$$

$$E_{\pm 2} = \pm\sqrt{10 - a^2 - b^2 + \sqrt{36 + 12\,a^2 + a^4 - 36\,b^2}}\,.$$

♠ $N = 2K$ **and** $N = 2K + 1$ **are similar:**

$E_0 = 0$ plus pairs $E_n = -E_{-n} = \sqrt{s}$ with $n = 1, 2,$

$$s^K + P_{K-1}(A, B, \ldots)\, s^{K-1} + P_{K-2}(A, B, \ldots)\, s^{K-2} + \ldots = 0$$

♠ **$N = 2K$ and $N = 2K+1$ are similar:**

$E_0 = 0$ plus pairs $E_n = -E_{-n} = \sqrt{s}$ with $n = 1, 2,$

$$s^K + P_{K-1}(A, B, \ldots)\, s^{K-1} + P_{K-2}(A, B, \ldots)\, s^{K-2} + \ldots = 0$$

♠ ♠ K polynomial equations for K unknowns:

$$P_{K-1}\left(A^{(EEP)}, B^{(EEP)}, \ldots\right) = 0,$$

$$P_{K-2}\left(A^{(EEP)}, B^{(DEEP)}, \ldots\right) = 0,$$

$$\ldots$$

$$P_0\left(A^{(EEP)}, B^{(EEP)}, \ldots\right) = 0\,.$$

♠ ♠ ♠ solution: Gröbner

♣ $N = 4$, quadratic, solvable:

$$A + 2\,B = 10, \qquad (3 + B)^2 = 9\,A$$

solution $A = 64$ and $B = -27$ is spurious,

the cusp is unique: $A^{(EEP)} = 4$ and $B^{(EEP)} = 3$.

♣ ♣ $N = 5$, quadratic, solvable:

inequalitites define **all the domain $\mathcal{D}$:**

$10 \geq A + B$ (circumscribed simplex),

$$36 + 12\,A + A^2 \geq 36\,B \qquad [B_{max} = B_{max}(A) = \text{parabola}]$$

$(8 + B)^2 \geq (32 - 2\,B)\,A \qquad [A_{max} = A_{max}(B)].$

$N = 6$

$$\det \begin{bmatrix} 5-E & c & 0 & 0 & 0 & 0 \\ -c & 3-E & b & 0 & 0 & 0 \\ 0 & -b & 1-E & a & 0 & 0 \\ 0 & 0 & -a & -1-E & b & 0 \\ 0 & 0 & 0 & -b & -3-E & c \\ 0 & 0 & 0 & 0 & -c & -5-E \end{bmatrix} = 0$$

$$E^6+\left(2\,b^2-35+a^2+2\,c^2\right)E^4+$$

$$+\left(-34\,a^2+2\,b^2c^2+28\,c^2+b^4+2\,c^2a^2+c^4-44\,b^2+259\right)E^2+$$

$$+a^2c^4+225\,a^2+30\,c^2a^2-225-10\,b^2c^2-25\,b^4-30\,c^2-c^4-150\,b^2=0$$

$$416\,C^4+20909\,C^3+22505\,C^2+28734375\,C-48828125=0$$

$$A^{(EEP)}=9\,,\quad B^{(EEP)}=8\,,\quad C^{(EEP)}=5\,,\qquad\qquad N=6,$$

there exist two further real roots: $C_-=-65.80360706$ (spurious) and $C_+=1.693394621$ such that B is negative:

$$22156250\,B_+ + 2912\,{C_+}^3 + 1446363\,{C_+}^2 + 820546875 + 9654410\,C_+ = 0$$

$N = 7$

$$A^{(EEP)} = 12\,, \quad B^{(EEP)} = 10\,, \quad C^{(EEP)} = 6\,,$$

the only positive root $C_+ = 68.24318125$ gives the negative $B = 28 - 3\,C$.

spurious (negative) also:

one of the two roots $C_\pm = 27 \pm 9\,\sqrt{21}$ of $C^2 - 54\,C = 972$

and *both* the roots $-354 \pm 60\,\sqrt{34}$ of $C^2 + 708\,C + 2916 = 0$

$N = 8$

$\mathcal{D}^{(8)}$ circumscribed by the simplex

$$A + 2\,B + 2\,C + 2\,D = 84\,.$$

quadratic, cubic and quartic polynomial equations $P_2(A, B, C, D) = 0$, $P_1(A, B, C, D) = 0$ and $P_0(A, B, C, D) = 0$

containing 13, 19 and 20 individual terms, respectively

reduced to 9-terms in P_2,

$$1974+(B+C+D)^2+2\,AD+2\,BD+2\,AC = 83\,A+142\,B+70\,C-50\,D$$

etc.

Groebner polynomial:

$$314432\,D^{17} - 5932158016\,D^{16} + 4574211144896\,D^{15} +$$

$$+3133529909492864\,D^{14} + 917318495163561932\,D^{13} + \ldots$$

$$+ \ldots + 2353267541018244399368002288069050 73\,D^{2} -$$

$$-45376227941462117981555289702903979 7\,D +$$

$$+153712881941946532798614648361265167 = 0$$

gives the unique, closed solution

$$A^{(EEP)} = 16\,, \quad B^{(EEP)} = 15\,, \quad C^{(EEP)} = 12\,, \quad D^{(EEP)} = 7\,, \qquad N = 8\,.$$

It possesses seven other real and positive roots D. Three other are real but manifestly spurious, $-203.9147095, -156.6667001, -55.49992441$. For the remaining four roots $0.4192854385, 5.354156128, 1354.675195$ and 18028.16789 we have indirectly proved spuriosity again. For example, A given by the rule $\alpha \times A =$ (a polynomial in D of 16th degree) where the number of digits in α exceeds one hundred.

$N = 9$

$$14745600 - 7372800\,A + \ldots + (-2\,C + 220 - 2\,B - 2\,A - 2\,D)\,s^4 - s^5 = 0$$

$$A^{(EEP)} = 20\,,\quad B^{(EEP)} = 18\,,\quad C^{(EEP)} = 14\,,\quad D^{(EEP)} = 8\,,\qquad N = 9\,.$$

E.

EXTRAPOLATIONS

Extrapolation to all even $N = 2K$

Cicrcumscribed simplex

$$A + 2\,(B + C + \ldots + Z) = \frac{4\,K^3 - K}{3}$$

or ellipsoid,

$$a^2 + 2\,b^2 + \ldots + 2\,z^2 \leq \frac{4\,K^3 - K}{3}\,.$$

Test by insertion performed for the resulting cusps:

$$A^{(EEP)} = K^2,\ B^{(EEP)} = K^2 - 1^2,\ C^{(EEP)} = K^2 - 2^2,\ D^{(EEP)} = K^2 - 3^2,\ \ldots\,.$$

i.e., $a^{(EEP)} = \pm K,\ \ b^{(EEP)} = \pm\sqrt{K^2 - 1}$ etc.

Extrapolation to all odd $N = 2M + 1$

$$A^{(EEP)} = M(M+1), \quad B^{(EEP)} = M(M+1) - 1 \cdot 2 = M(M+1) - 2,$$

$$C^{(EEP)} = M(M+1) - 2 \cdot 3, \quad D^{(EEP)} = M(M+1) - 3 \cdot 4, \; \ldots \,.$$

$$A + B + C + D + \ldots + Z = \frac{2\,M^3 + 3\,M^2 + M}{3}$$

$$a^2 + b^2 + \ldots + z^2 \leq \frac{2\,M^3 + 3\,M^2 + M}{3}$$

intersections at 2^M EEP points

$a^{(EEP)} = \pm\sqrt{M(M+1)}$, $b^{(EEP)} = \pm\sqrt{M(M+1) - 2}$ etc.

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All the results D and E are freshly published:

Miloslav Znojil,

Maximal couplings in PT-symmetric chain-models with the real

spectrum of energies

J. Phys. A: Math. Theor. 40 (2007) 4863 - 4875.

(math-ph/0703070).

F.

FINE-TUNING

F. I. Reparametrization

a routine return to the self-adjointness:

$\langle \phi \,|\, \psi \rangle \to \langle \phi \,|\, \Theta \,|\, \psi \rangle$, $\langle x \,|\, \Theta \,|\, x' \rangle \neq 0$

couplings

$$g_k^2 = G_k^{(N)} \left(1 - \gamma_k^{(N)}\right) , \qquad G_k^{(N)} = k\,(N-k)\,, \qquad \gamma_k^{(N)} \in (0,1)$$

task: explore the "physical" domain $\mathcal{D}$

Two-by-two case

$$H^{(2)} = \begin{pmatrix} -1 & \sqrt{1-\alpha} \\ -\sqrt{1-\alpha} & 1 \end{pmatrix}, \qquad \alpha \in (0,1)$$

$$E_{\pm}^{(2)} = \pm\sqrt{\alpha}$$

$$\mathcal{D}^{(2)}(\alpha) \; \equiv \; (0,1)$$

Three-by-three model

$$H^{(3)} = \begin{pmatrix} -2 & \sqrt{2-2\,\alpha} & 0 \\ -\sqrt{2-2\,\alpha} & 0 & \sqrt{2-2\,\alpha} \\ 0 & -\sqrt{2-2\,\alpha} & 2 \end{pmatrix}, \qquad \alpha \in (0,1)$$

$E_0^{(3)} = 0$ and $E_\pm^{(3)} = \pm 2\sqrt{\alpha}$

$\mathcal{D}^{(3)}(\alpha) \ \equiv \ (0,1)$

Four-by-four model

$$H^{(4)} = \begin{pmatrix} -3 & \sqrt{3-3\,\beta} & 0 & 0 \\ -\sqrt{3-3\,\beta} & -1 & 2\,\sqrt{1-\alpha} & 0 \\ 0 & -2\,\sqrt{1-\alpha} & 1 & \sqrt{3-3\,\beta} \\ 0 & 0 & -\sqrt{3-3\,\beta} & 3 \end{pmatrix}, \quad \alpha\,,\beta \in (0,1)\,.$$

$$s^2 - (6\,\beta + 4\,\alpha)\,s - 36\,\beta + 36\,\alpha + 9\,\beta^2 = 0$$

$$s_\pm = 3\,\beta + 2\,\alpha \pm 2\,\sqrt{3\,\beta\,\alpha + \alpha^2 + 9\,\beta - 9\,\alpha}\,.$$

we must guarantee that $s_\pm \geq 0$

(a) reality: the curve of the minimal β,

$$\beta \geq \beta_{minimal} = \frac{9\,\alpha - \alpha^2}{9 + 3\,\alpha}\,, \qquad \alpha \in (0,1)\,.$$

(b) $s_- \geq 0$: a minimum of α,

$$\alpha \geq \alpha_{minimal} = \beta - \frac{\beta^2}{4}\,, \qquad \beta \in (0,1)\,.$$

Five-by-five model

$$H^{(5)} = \begin{pmatrix} -4 & 2\sqrt{1-\beta} & 0 & 0 & 0 \\ -2\sqrt{1-\beta} & -2 & \sqrt{6-6\,\alpha} & 0 & 0 \\ 0 & -\sqrt{6-6\,\alpha} & 0 & \sqrt{6-6\,\alpha} & 0 \\ 0 & 0 & -\sqrt{6-6\,\alpha} & 2 & 2\sqrt{1-\beta} \\ 0 & 0 & 0 & -2\sqrt{1-\beta} & 4 \end{pmatrix}$$

$$s^2 - P_1^{(5)}(g_1, g_2)\, s + P_2^{(5)}(g_1, g_2) = 0$$

$$P_1^{(5)}(g_1, g_2) = 8\,\beta + 12\,\alpha\,, \qquad P_2^{(5)}(g_1, g_2) = 48\,\alpha\,\beta - 144\,\beta + 144\,\alpha + 16\,\beta^2\,.$$

$$P_2 \geq 0\,, \qquad\qquad P_1^2 - 4\,P_2 \geq 0\,.$$

(b) $\alpha \to A$, $\alpha = \alpha(A) = \beta + A\,\beta^2$, $A \geq -4/(3\,\beta + 9)$ in the $A - \beta$ plane

(c) $\beta \to B$ with $\beta = \beta(B) = \alpha + B\,\alpha^2$, $B \geq -1/4$

(d) $\beta = \beta[A] = 2\,\alpha/(1 + \sqrt{1 + 4\,\alpha\,A})$ etc.

Six-by-six model

$$g_1 = c = \sqrt{5\,(1-\gamma)}\,, \quad g_2 = b = 2\,\sqrt{2\,(1-\beta)}\,, \quad g_3 = a = 3\,\sqrt{1-\alpha}$$

$$H^{(6)} = \begin{bmatrix} -5 & g_1 & 0 & 0 & 0 & 0 \\ -g_1 & -3 & g_2 & 0 & 0 & 0 \\ 0 & -g_2 & -1 & g_3 & 0 & 0 \\ 0 & 0 & -g_3 & 1 & g_2 & 0 \\ 0 & 0 & 0 & -g_2 & 3 & g_1 \\ 0 & 0 & 0 & 0 & -g_1 & 5 \end{bmatrix}.$$

$$\det\left(H^{(6)} - E\,I\right) = s^3 - 3\,P\,s^2 + 3\,Q\,s - R = 0\,, \qquad s = E^2\,.$$

$$P = -\left(a^2 + 2\,b^2 + 2\,c^2 - 35\right)/3$$

$$3\,Q = b^4 + 2\,c^2a^2 - 44\,b^2 + 28\,c^2 - 34\,a^2 + c^4 + 259 + 2\,b^2c^2$$

$$-R = a^2c^4 - 10\,b^2c^2 + 30\,c^2a^2 + 225\,a^2 - 30\,c^2 - c^4 - 25\,b^4 - 225 - 150\,b^2$$

G.

Summary

Remember: we have elementary matrix models where the energies can get complex sometimes

- "user-friendly" tridiagonal $H^{(N)}$;
- all types of degeneracies of the energy pairs (followed by their $\mathcal{PT}-$symmetry-related complexifications) at all dimensions;
- *precisely* the necessary number of parameters;
- exact solvability
- tunable scenarios of "the first" or "instability" complexification