Solvable relativistic quantum dots with vibrational spectra

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For Klein-Gordon equation a consistent physical interpretation of wave functions is reviewed as based on a proper modification of the scalar product in Hilbert space. Bound states are then studied in a deep-square-well model where spectrum is roughly equidistant and where a fine-tuning of the levels is mediated by $\mathcal{PT}-$symmetric interactions (composed of imaginary delta functions) which mimic creation/annihilation processes.

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1 Klein-Gordon equation

1.1 σ_3-pseudo-Hermitian Feshbach-Villars Hamiltonian

As long as the most common relativistic Klein-Gordon (KG) operators are partial differential operators of the second order with respect to time, the time evolution of the wave functions $\Psi^{(KG)}(x,t)$ must be studied together with their first time derivatives $i\,\partial_t\,\Psi^{(KG)}(x,t)$. After the routine Fourier transformation we often arrive at the Feshbach-Villars (FV, [1]) non-Hermitian eigenvalue problem

$$\hat{H}^{(FV)}\,|\psi\rangle = E\,|\psi\rangle\,, \qquad \hat{H}^{(FV)} = \left(\begin{array}{cc} 0 & \hat{h}^{(KG)} \\ 1 & 0 \end{array} \right) \qquad (1)$$

where the two wave-function components [marked by the curly kets, viz., $|D\}$ (="down component") and $|U\}$ (="up component")] form a doublet,

$$|\psi\rangle = \left(\begin{array}{c} |U\} \\ |D\} \end{array} \right) \ \in\ \mathcal{H} = \mathcal{H}_{(c)} \oplus \mathcal{H}_{(c)}\,.$$

For the description of the bound states in one dimension we choose $\mathcal{H}_{(c)} = L_2(I\!\!R)$ and imagine that the two-by-two partitioning in (1) allows us to extract $|U\} = E\,|D\}$ and to replace our Klein-Gordon equation by its reduced, Schrödinger-like equivalent form

$$\hat{h}^{(KG)}\,|D_n\} = \varepsilon_n\,|D_n\}, \qquad |D_n\} \in \mathcal{H}_{(c)}, \qquad n = 1,2,\ldots \qquad (2)$$

with squared energy E^2 abbreviated as ε and with the "large" Hilbert space $\mathcal{H}$ of kets $|\psi\rangle$ reduced to the "smaller" Hilbert space $\mathcal{H}_{(c)}$ of the **c**urly-ket "down" components $|D_n\}$ [2].

1.2 Biorthogonal bases

The construction of the "right" eigenkets $|D_n\}$ in eq. (2) does not provide enough information about $\hat{h}^{(KG)}$ itself, due to its asymmetry which makes $[\hat{h}^{(KG)}]^\dagger$ a *different* operator. The parallel Schrödinger-type problem $\left[\hat{h}^{(KG)}\right]^\dagger |L_n\}\} = \kappa_n^2 \, |L_n\}\}$ generates, therefore, *different* eigenkets marked by the double curly ket symbol.

The latter sequence may be read as the left eigenvectors of our original operator $\hat{h}^{(KG)}$. Its elements are related to the same (real) eigenvalues $\varepsilon_n \equiv \kappa_n^2$ so that we have to keep the whole pair of the Schrödinger-like bound-state equations in mind,

$$\hat{h}^{(KG)} \, |D_n\} = \kappa_n^2 \, |D_n\}, \qquad \{\{L_n| \, \hat{h}^{(KG)} = \kappa_n^2 \, \{\{L_n|, \qquad n = 1, 2, \ldots . \tag{3}$$

It is well known that the set of the bras $\{\{L_n|$ and kets $|D_n\}$ is bi-orthogonal [2],

$$\{\{L_m|D_n\} = 0 \quad \text{for} \quad m \neq n,$$

and that it forms, usually, a basis in the infinite-dimensional Hilbert space $\mathcal{H}_{(c)}$. Thus, once we manage to evaluate all the non-vanishing overlaps $\{\{L_n|D_n\} \equiv \varrho_n$ we may decompose the unit operator in $\mathcal{H}_{(c)}$,

$$I_{(c)} = \sum_{n=1}^{\infty} |D_n\} \, \frac{1}{\varrho_n} \, \{\{L_n| \, . \tag{4}$$

Equally easily we derive the bi-orthogonal spectral representation of

$$\hat{h}^{(KG)} = \sum_{n=1}^{\infty} |D_n\} \, \frac{\kappa_n^2}{\varrho_n} \, \{\{L_n| \, . \tag{5}$$

The overlaps ϱ_n need not be all of the same sign.

2 Relativistic observables

2.1 $\Theta-$quasi-Hermiticity

In the space $\mathcal{H} = \mathcal{H}_{(c)} \oplus \mathcal{H}_{(c)}$ of the eigenstates of $H^{(FV)}$ we have to consider the pair of conjugate equations

$$\hat{H}^{(FV)} \, |n^{(\pm)}\rangle = \pm\kappa_n \, |n^{(\pm)}\rangle, \qquad \langle\langle n^{(\pm)}| \, \hat{H}^{(FV)} = \pm\kappa_n \, \langle\langle n^{(\pm)}| \, . \tag{6}$$

Both the left and right eigenstates have the two-component structure,

$$|m^{(\pm)}\rangle\rangle = \left(\begin{array}{c} |L_m\}\} \\ \pm\kappa_m \, |L_m\}\} \end{array} \right), \qquad |n^{(\pm)}\rangle = \left(\begin{array}{c} \pm\kappa_n \, |D_n\} \\ |D_n\} \end{array} \right)$$

and form the bi-orthogonal set in the "bigger" space $\mathcal{H}$,

$$\langle\langle m^{(\nu)}|n^{(\nu\prime)}\rangle = \delta_{mn}\delta_{\nu\nu\prime} \cdot \mu_m^{(\nu)}, \qquad \mu_m^{(\pm)} = \pm 2\kappa_m \, \varrho_m \, , \qquad \nu, \, \nu\prime = \pm 1 \, .$$

It is expected to be complete,

$$I = \sum_{\tau=\pm1} \sum_{n=1}^{\infty} |n^{(\tau)}\rangle \, \frac{1}{\mu_n^{(\tau)}} \, \langle\langle n^{(\tau)}| \,, \qquad \mathcal{H} = \mathcal{H}_{(c)} \oplus \mathcal{H}_{(c)} \,. \tag{7}$$

and useful,

$$H^{(FV)} = \sum_{\tau=\pm1} \sum_{n=1}^{\infty} |n^{(\tau)}\rangle \, \frac{\tau \, \kappa_n}{\mu_n^{(\tau)}} \, \langle\langle n^{(\tau)}| = \sum_{n=1}^{\infty} \frac{\left(|n^{(+)}\rangle \, \langle\langle n^{(+)}|\right) + \left(|n^{(-)}\rangle \, \langle\langle n^{(-)}|\right)}{2 \, \varrho_n} \,.$$

Let us now assume that at a given $\hat{H}^{(FV)}$, equation

$$\left[\hat{H}^{(FV)}\right]^{\dagger} = \eta \, \hat{H}^{(FV)} \, \eta^{-1} \tag{8}$$

possesses a positive and Hermitian solution $\eta_+ = \Theta > 0$. Such an operator may play the role of *metric* and induces the following specific scalar product in $\mathcal{H}$,

$$(|\psi_1\rangle \odot |\psi_2\rangle) = \langle\psi_1| \, \Theta \, |\psi_2\rangle = \langle\psi_1|\psi_2\rangle_{(physical)}, \qquad |\psi_1\rangle \in \mathcal{H}, \quad |\psi_2\rangle \in \mathcal{H} \,. \tag{9}$$

This product generates the norm, $||\psi|| = \sqrt{\langle\psi|\psi\rangle_{(physical)}}$. In terms of the later product and metric we may call all the operators A with the property $A^{\dagger} = \Theta \, A \, \Theta^{-1}$ *quasi-Hermitian* and treat them as observables (see [3] for a deeper outline of some more sophisticated mathematical details). Indeed, we have

$$(|\psi_1\rangle \odot |A \, \psi_2\rangle) \equiv (|A \, \psi_1\rangle \odot |\psi_2\rangle) \tag{10}$$

so that the probabilistic expectation values $\langle\psi| \, A \, |\psi\rangle_{(physical)}$ are mathematically unambigously defined.

2.2 Explicit constructions of the metric Θ

We have seen that in the language of pure mathematics, our present innovation admitting the non-Hermiticity $\hat{h}^{(KG)} \neq [\hat{h}^{(KG)}]^{\dagger} = \mathcal{P} \, \hat{h}^{(KG)} \, \mathcal{P}$ cannot lead to any real complications. Now it remains for us to assign a consistent *physical* meaning to all our relativistic bound states of section 3. In the other words, once we have solved the underlying auxiliary reduced eq. (2), we still have to select or construct a suitable physical metric Θ connected with the Feshbach-Villars Hamiltonians $\hat{H}^{(FV)}$ by the metric-operator definition (8).

Within the framework of our present class of models, the construction of the necessary (family of) metrics Θ is straightforward. In the first step we have to recollect that our trigonometric formulae (15) for the vectors $|D_n\}$ define, in closed form, all the necessary Hilbert-space vectors $|n^{(\pm)}\rangle$ as well as, *mutatis mutandis*, all their partners $|n^{(\pm)}\rangle\rangle$ (obtainable via the replacement of our auxiliary $\hat{h}^{(KG)}$ by its Hermitian conjugate $[\hat{h}^{(KG)}]^{\dagger}$ which is trivial).

In the second step we find out that the evaluation of the overlaps ϱ_n remains feasible and requires merely some symbolic-manipulation programming in more complicated cases. This follows from the fact that all our wave functions $|D_n\}$ and $|D_n\}\}$ are piecewise trigonometric.

In the final step we may employ an ansatz

$$\Theta = \sum_{\tau,\tau\prime=\pm 1} \sum_{m,n=1}^{\infty} |n^{(\tau)}\rangle\rangle\, M_{nm}^{(\tau\tau\prime)}\, \langle\langle m^{(\tau\prime)}|\,,$$

the backward insertion of which in (8) gives the condition

$$\tau\,\kappa_n\, M_{nm}^{(\tau\tau\prime)} = M_{nm}^{(\tau\tau\prime)}\,\tau\prime\,\kappa_m$$

with the set of solutions $M_{nm}^{(\tau\tau\prime)} = \omega_n^{(\tau)}\delta_{nm}\,\delta_{(\tau\tau\prime)}$ numbered by the free parameters $\vec{\omega}^{(\pm)}$. The Hermiticity and positivity constraints restrict the freedom of the choice of both the optional sequences $\vec{\omega}^{(\pm)}$ to the real and positive values, $\omega_n^{(\pm)} > 0$. *Vice versa*, *any* choice of the latter two sequences defines an eligible operator of the metric

$$\Theta = \Theta_{\vec{\omega}^{(\pm)}} = \sum_{\tau=\pm 1} \sum_{n=1}^{\infty} |n^{(\tau)}\rangle\rangle\, \omega_n^{(\tau)}\, \langle\langle n^{(\tau)}|\,. \qquad (11)$$

Its inverse

$$\Theta^{-1} = \sum_{\tau=\pm 1} \sum_{n=1}^{\infty} |n^{(\tau)}\rangle\, \frac{1}{\omega_n^{(\tau)}|\,\mu_n^{(\tau)}|^{\,2}}\, \langle n^{(\tau)}| \qquad (12)$$

is similar in its form. We may conclude that in terms of the metric Θ, our bound-state wave functions constructed in section 3 acquire the standard probabilistic interpretation.

3 The models with complex delta-interactions

Potentials $V(x)$ will be *complex* (otherwise we would return to $\theta = I$) while the energies are expected real. Our requirement of the $\mathcal{PT}-$symmetry

$$\mathcal{T}\,\hat{h}^{(KG)}\,\mathcal{T} \equiv \left[-\frac{d^2}{dx^2} + m_0^2 + V(x)\right]^{\dagger} = \mathcal{P}\,\left[-\frac{d^2}{dx^2} + m_0^2 + V(x)\right]\,\mathcal{P} \qquad (13)$$

demands that while the coordinate $x \in I\!\!R$ is real, the real and imaginary parts of the scalar potential are spatially symmetric and antisymmetric, respectively, $V(x) = V^*(-x)$. Further possible generalizations (say, with complex contours of x in the relativistic case, etc) will not be discussed here. Still, the class of models compatible with eq. (13) remains fairly broad. In this paper we shall restrict it significantly in a way inspired by the success of several non-relativistic studies of $\mathcal{PT}-$symmetric models with point interactions [4].

Making an explicit choice among all the eligible point-interaction candidates $V(x)$ we shall follow the encouraging experience which we made in our paper [5] where we combined the infinitely deep square-well real part of the potential with the delta-function formula for its imaginary part. Thus, once we put $V(x) = \infty$ for all $x \notin (-1,1)$ we shall postulate here that

$$V(x) = \sum_{\ell=1}^{\mathcal{L}} \left[i\,\xi_\ell\,\delta\,(x-a_\ell) - i\,\xi_\ell\,\delta\,(x+a_\ell) \right]\,, \qquad x \in (-1,1) \tag{14}$$

at real couplings ξ_ℓ and ordered points $0 < a_1 < a_2 < \ldots < a_{\mathcal{L}-1} < a_{\mathcal{L}} < 1$.

3.1 Wave functions

The key advantage of our choice of $V(x)$ in eq. (14) is that the $\mathcal{PT}-$symmetrically normalized coordinate representation $\psi(x) = \psi^*(-x)$ of the ket $|D\}$ in eq. (2) remains piecewise trigonometric. At each real and positive bound-state energy $\varepsilon = \kappa^2$ we shall have

$$\psi(x) = \begin{cases} \psi_L^{(\mathcal{L})}(x) = (\alpha_{\mathcal{L}} - i\,\beta_{\mathcal{L}})\,\sin\kappa(1+x), & x \in (-1,-a_{\mathcal{L}}), \\ \psi_L^{(\ell)}(x) = (\alpha_\ell - i\,\beta_\ell)\,\sin\kappa(a_{\ell+1}+x) + (\gamma_\ell - i\,\delta_\ell)\,\cos\kappa(a_{\ell+1}+x), & \\ & x \in (-a_{\ell+1},-a_\ell), \\ \psi_C^{(0)}(x) = \mu\,\cos\kappa x + i\,\nu\,\sin\kappa x, & x \in (-a_1,a_1), \\ \psi_R^{(\ell)}(x) = (\alpha_\ell + i\,\beta_\ell)\,\sin\kappa(a_{\ell+1}-x) + (\gamma_\ell + i\,\delta_\ell)\,\cos\kappa(a_{\ell+1}-x), & \\ & x \in (a_\ell,a_{\ell+1}), \\ \psi_R^{(\mathcal{L})}(x) = (\alpha_{\mathcal{L}} + i\,\beta_{\mathcal{L}})\,\sin\kappa(1-x), & x \in (a_{\mathcal{L}},1), \quad 1 \le \ell < \mathcal{L}. \end{cases} \tag{15}$$

This formula simplifies not only the continuity conditions

$$\begin{array}{c} \psi_L^{(\ell-1)}(-a_\ell) = \psi_L^{(\ell)}(-a_\ell), \quad \ell = \mathcal{L}, \mathcal{L}-1, \ldots, 2, \\ \psi_C^{(0)}(-a_1) = \psi_L^{(1)}(-a_1), \qquad\qquad \psi_R^{(1)}(a_1) = \psi_C^{(0)}(a_1), \\ \psi_R^{(\ell+1)}(a_{\ell+1}) = \psi_R^{(\ell)}(a_{\ell+1}), \quad \ell = 1, 2, \ldots, \mathcal{L}-1, \end{array} \tag{16}$$

but also the differentiation,

$$\frac{1}{\kappa}\psi'(x) = \begin{cases} (\alpha_{\mathcal{L}} - i\,\beta_{\mathcal{L}})\,\cos\kappa(1+x), & x \in (-1,-a_{\mathcal{L}}), \\ (\alpha_\ell - i\,\beta_\ell)\,\cos\kappa(a_{\ell+1}+x) - (\gamma_\ell - i\,\delta_\ell)\,\sin\kappa(a_{\ell+1}+x), & \\ & x \in (-a_{\ell+1},-a_\ell), \\ -\mu\,\sin\kappa x + i\,\nu\,\cos\kappa x, & x \in (-a_1,a_1), \\ -(\alpha_\ell + i\,\beta_\ell)\,\cos\kappa(a_{\ell+1}-x) + (\gamma_\ell + i\,\delta_\ell)\,\sin\kappa(a_{\ell+1}-x), & \\ & x \in (a_\ell,a_{\ell+1}), \\ -(\alpha_{\mathcal{L}} + i\,\beta_{\mathcal{L}})\,\cos\kappa(1-x), & x \in (a_{\mathcal{L}},1), \quad \ell = 1, 2, \ldots, \mathcal{L}-1\,. \end{cases} \tag{17}$$

All this enters the definition of the action of the delta functions,

$$\begin{array}{c} \left[\psi_L^{(\ell-1)}(-a_\ell)\right]' - \left[\psi_L^{(\ell)}(-a_\ell)\right]' = -i\xi_\ell\,\psi_L^{(\ell)}(-a_\ell), \quad \ell = \mathcal{L}, \mathcal{L}-1, \ldots, 2, \\ \left[\psi_C^{(0)}(-a_1)\right]' - \left[\psi_L^{(1)}(-a_1)\right]' = -i\xi_1\,\psi_C^{(0)}(-a_1), \\ \left[\psi_R^{(1)}(a_1)\right]' - \left[\psi_C^{(0)}(a_1)\right]' = i\xi_1\,\psi_C^{(0)}(a_1), \\ \left[\psi_R^{(\ell+1)}(a_{\ell+1})\right]' - \left[\psi_R^{(\ell)}(a_{\ell+1})\right]' = i\xi_{\ell+1}\,\psi_R^{(\ell)}(a_{\ell+1}), \quad \ell = 1, 2, \ldots, \mathcal{L}-1, \end{array} \tag{18}$$

After the insertion of the ansatzs (15) and (17), the set of formulae (16) and (18) may be read as a homogeneous linear algebraic system of $4\mathcal{L}$ equations for the $4\mathcal{L}$ unknown wave-function coefficients $\alpha_{\mathcal{L}}$, $\beta_{\mathcal{L}}$, $\ldots$, ν. The secular determinant $\mathcal{D}(\kappa)$ of this system must vanish so that the not too complicated transcendental equation

$$\mathcal{D}(\kappa) = 0 \tag{19}$$

determines finally the set of the bound-state roots $\kappa = \kappa_n$ at $n = 1, 2, \ldots$.

3.2 Energies at the simplest choice of $\mathcal{L} = 1$

Let us pick up $\mathcal{L} = 1$ and check how the method works. Firstly, potential (14) degenerates to the most elementary double-well model with the single coupling $\xi_1 = \xi$ and one displacement $a_1 = a$. Out of the related eight real constraints (16) and (18) only four are independent and define the four real coefficients $\alpha_1 = \alpha$, $\beta_1 = \beta$ and μ and ν as an eigenvector of a four-by-four matrix with the secular determinant [6]

$$\mathcal{D}(\kappa) = -\frac{1}{2}\left\{\sin 2\kappa + \frac{\xi^2}{\kappa^2}\,\sin 2\kappa a\,\cdot \sin^2[\kappa(1-a)]\right\}\,. \tag{20}$$

Numerically, the first term would give us the well-known square-well spectrum at $\xi = 0$, the completeness of which is controlled by the Sturm-Liouville oscillation theory [7]. As long as all the roots $\kappa_n = \kappa_n(\xi)$ are smooth and real functions of ξ at the smallest couplings, $\kappa_n(\xi) \approx n\pi/2 + \mathcal{O}(\xi^2/n)$, our explicit construction confirms the general mathematical prediction [8] that the influence of the non-Hermiticity will be most pronounced at the lowest part of the spectrum.

3.3 The next choice of $\mathcal{L} = 2$

On the quadruple-well potential (14) with $\mathcal{L} = 2$ we may verify a smoothness of its degeneracy to the more elementary double well of previous subsection. Working now with the two couplings ξ_1 and ξ_2 we may shorten our notation for the points of interactions ($a_1 = a$, $a_2 = b$), drop the two redundant subscripts ($\gamma_1 = \gamma$, $\delta_1 = \delta$) and evaluate the eight-dimensional matrix of the system for our eight coefficients in eq. (15). We reveal that the elimination of some of them is trivial [$\gamma = \alpha_2 \sin\kappa(1-b)$, $\delta = \beta_2 \sin\kappa(1-b)$] or at least sufficiently easy [$\alpha_1 = \alpha_1(\alpha_2, \beta_2)$, $\beta_1 = \beta_1(\alpha_2, \beta_2)$]. We end up with a four-by-four matrix problem, simplified further by trigonometric identities. We get the secular determinant

$$\mathcal{D}(\kappa) = \mathcal{D}_{(0)}(\kappa) + \mathcal{D}_{(\xi_1)}(\kappa) + \mathcal{D}_{(\xi_2)}(\kappa) + \mathcal{D}_{(\xi_1\xi_2)}(\kappa) \tag{21}$$

where

$$\mathcal{D}_{(0)}(\kappa) = -\frac{1}{2}\sin 2\kappa\,, \qquad \mathcal{D}_{(\xi_j)}(\kappa) = -\frac{\xi_j^2}{2\kappa^2}\,\sin 2\kappa a_j\,\cdot \sin^2[\kappa(1-a_j)], \quad j = 1, 2$$

while

$$\mathcal{D}_{(\xi_1\xi_2)}(\kappa) = -\left\{\frac{\xi_1\xi_2}{\kappa^2}\,\sin\,2\kappa a + \frac{\xi_1^2\xi_2^2}{\kappa^4}\,\sin^2[\kappa(b-a)]\right\}\sin^2[\kappa(1-b)]\,.$$

The derivation of this secular determinant remains feasible without symbolic manipulations on a computer. The function itself correctly degenerates to the previous $\mathcal{L} = 1$ formula in both the independent limits of $\xi_1 \to 0$ and $\xi_2 \to 0$.

3.4 Simplifications at the rational a_j

Let us return to the secular eq. (20) with $\mathcal{L} = 1$ and choose $a = 1/2$ [6]. This leads to a factorization of $\mathcal{D}(\kappa)$ and to the pair of the eigenvalue conditions

$$\cos\kappa_{2m-1} = \frac{\xi^2}{\xi^2 - 4\kappa_{2m-1}^2}, \qquad \sin\kappa_{2m} = 0, \qquad m = 1, 2, \ldots \tag{22}$$

with the second series of equations being exactly solvable, $\kappa_{2m} = m\pi$.

At the next choice of $a = 1/3$ we factorize eq. (20) in the similar manner and get the series of the $\xi-$dependent roots specified by the implicit definitions

$$\cos\frac{4}{3}\kappa_p = \frac{\xi^2 + 2\kappa_p^2}{\xi^2 - 4\kappa_p^2}, \qquad p = 1, 2, 4, 5, 7, 8, 10, \ldots \tag{23}$$

complemented by the closed formula for all the skipped roots of the factor $\sin 2\kappa/3$ which remain $\xi-$independent and read $\kappa_{3m} = 3m\pi/2$ with $m = 1, 2, \ldots$. The regularity of such a pattern of the $\xi-$independent roots is easily prolonged to the decreasing sequence of a with $\kappa_{4m} = 2m\pi$ at $a = 1/4$ and all $m = 1, 2, \ldots$, etc.

The less elementary composite choice of $a = 2/3$ may be observed to give the same factor as at $a = 1/3$ and, hence, the same $\xi-$independent series of the roots $\kappa_{3m} = 3m\pi/2$ with $m = 1, 2, \ldots$. The implicit formula for the remaining roots is a slightly more complicated quadratic equation in the trigonometric unknown $X = \cos 2\kappa/3$,

$$\left(4\,\kappa^2 - \xi^2\right)\, X^2 + \xi^2\, X - \kappa^2 = 0\,. \tag{24}$$

Its trigonometric part X may be eliminated in the form resembling eq. (23).

One of the important consequences of the existence of the elementary formulae for the rational a is that they allow us to perform an elementary analysis of the qualitative features of the $n-$th root κ_n during the growth of the strength ξ of the non-Hermiticity. During such an analysis one discovers that these levels are either "robust" (marked by a superscript, $\kappa_n^{(R)}$, and remaining real for all ξ) or "fragile" (such a $\kappa_n^{(F)}$ will merge with another $\kappa_m^{(F)}$ at a "critical" $\xi_{n,m}^{(C)}$ while the pair will complexify beyind this "exceptional" [9] point). For illustration let us display the regularity of the pattern in the simplest spectra,

$$\kappa_1^{(F)},\ \kappa_2^{(R)},\ \kappa_3^{(F)},\ \kappa_4^{(R)},\ \kappa_5^{(F)},\ \kappa_6^{(R)},\ \ldots, \qquad a = 1/2$$

$$\kappa_1^{(F)},\ \kappa_2^{(F)},\ \kappa_3^{(R)},\ \kappa_4^{(F)},\ \kappa_5^{(F)},\ \kappa_6^{(R)},\ \ldots, \qquad a = 1/3$$

$$\kappa_1^{(F)},\ \kappa_2^{(F)},\ \kappa_3^{(R)},\ \kappa_4^{(R)},\ \kappa_5^{(R)},\ \kappa_6^{(F)},\ \kappa_7^{(F)},\ \kappa_8^{(R)},\ \kappa_9^{(R)},\ \kappa_{10}^{(R)},\ \kappa_{11}^{(F)},\ \ldots, \qquad a = 1/4$$

etc. An extension of these observations to the further few not too complicated rational distances a is rather routine and may be left to the interested reader.

4 Summary: Physical interpretation

As we already noticed, our reduced differential eq. (2) is formally similar to the standard Schrödinger equation. In particular, the role of the nonrelativistic Hamiltonian is played by our auxiliary operator $\hat{h}^{(KG)}$. At the same time, as long as $\hat{h}^{(KG)}$ acts on the mere component-subspace $\mathcal{H}_{(c)}$, several subtle differences between the physical meaning of $\hat{h}^{(KG)}$ in the non-relativistic and relativistic Quantum Mechanics must be underlined.

We may conclude that whenever we decide to treat $\mathcal{H}$ as a Hilbert space of states endowed with the particular metric Θ, all the operators A which prove quasi-Hermitian can be interpreted as observables.

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