Eight exactly solvable complex potentials in Bender - Boettcher quantum mechanics

Miloslav Znojil

Ústav jaderné fyziky AV ČR, 250 68 Řež, Czech Republic
homepage: http://gemma.ujf.cas.cz/~ znojil
e-mail: znojil@ujf.cas.cz

Abstract

We review the recent progress in the search for complex solvable potentials which exhibit a $\mathcal{PT}$ symmetry and shape invariance and lead to the *real* bound-state energy spectra.

PACS 03.65.Ge, 03.65.Fd

XX. Winter School "Geometry and physics", January 15 - 22, 2000, Srní, Šumava, Czech Republic,
version of March 27, 2000, Latex file srn.tex

1 Exactly solvable real potentials

This text offers a preliminary synthesis of recent, not yet published results on *non-Hermitian* Schrödinger bound state problem. Our review concerns the *complex* exactly solvable models but it will parallel very closely the existing classification of the exactly solvable *real* potentials. For definitness, we shall refer to page 296 of the review paper [1] which lists the real solvable potentials and characterizes them globally by their so called shape invariance. This list decays in two separate families generating the Laguerre and Jacobi polynomial wave functions. For convenience, we shall denote these families by symbols $\mathcal{L}$ and $\mathcal{J}$, respectively.

In the former family the first subset is to be understood as potentials $V(x)$ acting on the whole real axis, $x \in (-\infty, \infty)$. In a way quoting paper [1] as our key reference this subset can be agreed as containing just the most common harmonic oscillator $V^{(H)}(x) \sim x^2$ and the exponential Morse interaction $V^{(M)}(x)$. We shall denote this subset as $\mathcal{L}X$. The second subset $\mathcal{L}R$ of the former family consists of the two exceptional three-dimensional solvable models, viz., spiked harmonic $V^{(S)}(r) \sim r^2 + g/r^2$ and Kratzer-Coulomb $V^{(C)}(r) \sim e/r + g/r^2$. Both these forces depend on a coupling $g > g_{min}$ and are defined on the half axis of coordinates $r \in (0, \infty)$.

The other family $\mathcal{J}$ can be (and has been) conveniently split in the three separate subcategories, with two elements each. They are again distinguished by their specific ranges of coordinates. In the first subset $\mathcal{J}X$ the coordinates cover the whole real axis. Following the notation and terminology of ref. [1] we find there the Rosen Morse oscillator $V^{(RM)}(x)$ and the scarf-shaped potential $V^{(SS)}(x)$. Similarly, the respective Pöschl-Teller and Eckart-Hulthén forces $V^{(PT)}(x)$ and $V^{(EH)}(x)$ are names of the items which form the second subset $\mathcal{J}R$. A particle can also live on a finite interval within the framework of the third subset $\mathcal{J}Y$. We skip and ignore it completely here, due to its less immediate physical interpretation.

2 Complex potentials with $\mathcal{PT}$ symmetry

Within the slightly more general, non-Hermitian quantum mechanics with principles outlined by Bender and Boettcher in the recent series of their papers [2, 3] the current Hermiticity of a Hamiltonian

$$H = H^+$$

is tentatively being replaced by the condition

$$H = \mathcal{PT}\, H\, \mathcal{PT}\,. \tag{1}$$

Here, $\mathcal{P}$ changes the parity and the complex conjugation operator $\mathcal{T}$ transforms i to $-i$ and, in this way, mimics the time reversal. On a purely empirical basis the latter condition of $\mathcal{PT}$ symmetry often leads to the fully real and discrete spectrum. At the same time, it finds interesting interpretations in several phenomenological models [4]. As a consequence, the new formal framework of the so called $\mathcal{PT}$ symmetric quantum mechanics offers suddenly a number of open mathematical questions.

One of the earliest studies of the $\mathcal{PT}$ symmetric potentials by Caliceti et al [5] paid an exclusive attention to the cubic anharmonic oscillator. For the (sufficiently small) purely imaginary anharmonicities it delivered the first rigorous proof of the (at that time, quite puzzling) reality of the separate energies. The proof has been based on the Borel resummation of the Rayleigh-Schrödinger perturbation series. Its relevance has been appreciated in the subsequent numerical studies of the same particular system [6].

Another paper on a $\mathcal{PT}$ symmetric model by Buslaev and Grecchi [7] delivered a non-perturbative proof of the reality of the energies for a quartic polynomial force V. This proof has been based on a mind-boggling Fourier-transformation-mediated connection which implied a spectral equivalence between the complex V and its purely real image $\tilde{V}$. Renewed interest in similar interactions re-appeared in the literature only very recently [8].

The third, decisive encouragement for a more systematic interest in the $\mathcal{PT}$ symmetric systems has been found, by Bender and Boettcher, in the complexified one-dimensional harmonic oscillator itself [2]. Obviously, the difficult question of consequences of the condition (1) can be, in principle, most easily studied within the most transparent domain of the exactly solvable interactions. In this sense, the present paper just reviews the effort based on such a type of inspiration.

3 Complexified family $\mathcal{L}$ with $\mathcal{PT}$ symmetry

3.1 $\mathcal{L}X$

$\mathcal{PT}$ symmetrized models $V^{(H)}(x)$ and $V^{(M)}(x)$ have been described in refs. [2] and [9], respectively. The former force can be written in the fully general form

$$V^{(H)}(x) = \frac{1}{2}\,\omega^2 \left(x - \frac{2\,i\,\beta}{\omega}\right)^2 - \frac{\omega}{2} \tag{2}$$

with $\omega > 0$ and with any real β. This potential is shape invariant and in this sense it coincides, up to the replacement of symbols $b \to i\,\beta$, with the shifted oscillator listed as the first item in Table 4.1 of ref. [1]. The change of b leaves the spectrum unchanged. Wave functions become complexified by the same trivial substitution $b = i\,\beta$.

The bound state problem with the latter force

$$V^{(M)}(x) = -\omega^2 \exp(4ix) - D\,\exp(2ix)$$

is apparently much more complicated. Firstly, in quite an unusual manner, it is defined on the down-bent complex curve containing one free parameter $\varepsilon > 0$,

$$\mathcal{C} = \{x = v - iu \;||\; v \in (-\pi/2, \pi/2),\; u = u(v) = \ln(\varepsilon/\cos v)\}\,. \tag{3}$$

In terms of the two quantum numbers $q = \pm 1$ and $m = 0, 1, \ldots$ the sequence of the related energies reads

$$E^{(M)}_{(m,q)} = (2m + 1 \mp D/2\omega)^2$$

and is discussed in more detail in ref. [9].

A re-establishment of correspondence of eq. (2) to the usual real Morse force is instructive in showing that and how the new discrete spectrum is significantly richer. The comparison only requires a re-definition of couplings (say, $B = i\omega$ and $A = i(1 - D/2\omega)$ in the notation of ref. [1]) and a convenient re-scaling of the coordinate, $2\,i\,x \to -\alpha\,x$. Then, the transition to the formulae of ref. [1] is easy, admitted only for the energies with the positive "quasi-parity" $q = +1$ and within a restricted range of the principal quantum number, $m < A/\alpha$.

Beyond this correspondence, some of the other specific features of the new spectrum (e.g., an unavoided crossing of its levels) have not yet been interpreted satisfactorily. One of the possible lines of a new progress could be sought in the very similar single-exponential $\mathcal{PT}$ symmetric model of refs. [10] which proves solvable non-polynomially, in terms of the Bessel special functions.

3.2 $\mathcal{L}R$

Basic facts about the Laguerre-solvable and $\mathcal{PT}$ symmetrized spiked harmonic model

$$V^{(S)}(r) = r^2 + \frac{g}{r^2}, \qquad r = x - i\,\varepsilon, \qquad x \in I\!R, \quad \varepsilon > 0$$

can be found in the letter [11]. The presence of a spike $\sim g$ is shown there to induce just a fairly smooth change of the energy spectrum,

$$E^{(S)}_{(q,m,\ell)} = 4m + 2 - 2\,q\,\alpha, \qquad q\,\alpha = \pm\sqrt{\frac{1}{4} + g + \ell(\ell+1)}, \qquad m, \ell = 0, 1, \ldots \ \ldots .$$

Its levels depend on the angular momentum ℓ and are, therefore, numbered by the triplet of integers.

One has to replace ℓ by $\ell + (D-3)/2$ in the general dimension $D \neq 3$. Also, following exercise 17 on p. 442 in the Newton's book [12] the straightforward transition to $V^{(C)}(r)$ in the pertaining Schrödinger equation

$$\left[-\frac{d^2}{dr^2} + \frac{L(L+1)}{r^2} + i\,\frac{e^2}{r}\right] \psi(r) = E\,\psi(r) \tag{4}$$

can be performed easily [13]. In the $\mathcal{PT}$ symmetric case the purely imaginary charge must be used. This gives the positive spectrum

$$E^{(C)}_{(q,m,L)} = \frac{e^4}{[2m+1-q\,(2L+1)]^2}, \qquad m = 0, 1, \ldots$$

with the same interpretation of the angular-momentum-like $L = L(\ell, D, g)$ and with the same variability of $\ell = 0, 1 \ldots .$ and of the related centrifugal term in eq. (4).

4 Complexified family $\mathcal{J}$ with $\mathcal{PT}$ symmetry

4.1 $\mathcal{J}X$

In our present notation the $\mathcal{PT}$ symmetrized scarf-shaped model reads

$$V^{(SS)}(x) = \frac{-\beta^2 - A^2 - \mu\, A + i\,(2A+\mu)\,\beta\,\sinh\mu\,x}{\cosh^2\mu\,x}$$

and has a $\beta-$independent spectrum [14, 15]

$$E = -(A-\mu\,n)^2, \qquad n < A/\mu.$$

Its equivalence to the real case with $i\,\beta \to B$ resembles the above-mentioned harmonic oscillator example.

Transition to the second complexified shape invariant interaction of the Rosen-Morse type,

$$V^{(RM)}(x) = -\frac{A(A+1)}{\cosh^2 x} + 2\,i\,\gamma\,\frac{\sinh x}{\cosh x}, \qquad r \in I\!R$$

must be performed more carefully [15]. This asymptotically purely imaginary potential gives the more complicated spectrum

$$E_n^{(RM)} = -(A-\mu\,n)^2 + \gamma^2/(A-\mu\,n)^2, \qquad n < A.$$

One can really get puzzled by the latter innocent-looking, smooth and asymptotically vanishing potential. Its closer inspection recovers that it supports a ground state in the weak-coupling regime. The energy of this state can be arbitrarily large. Thus, for $\gamma = \mathcal{O}(\delta)$ and $A = \mathcal{O}(\delta^2)$ in $|V^{(RM)}(x)| < \delta^2$ we still have $E_0 = \mathcal{O}(1/\delta)$. This paradox offers one of the reasons why each separate solvable model with $\mathcal{PT}$ symmetry deserves particular attention and detailed analysis.

4.2 $\mathcal{J}R$, Eckart-Hulthén case

Moving to the last two shape invariant real interactions $V^{(EH)}(r)$ and $V^{(PT)}(r)$ of ref. [1] we notice that both these forces are, generically, strongly singular in the origin. This is their key difference from the previous two oscillators. In the first $\mathcal{PT}$ symmetrized singular model $V^{(EH)}(r)$ let us contemplate a purely imaginary part with variable strength $B = i\,\beta$,

$$V^{(EH)}(r) = \frac{A(A-1)}{\sinh^2 r} - 2\,i\,\beta\,\frac{\cosh r}{\sinh r}, \qquad A > 1/2.$$

This circumvents the conventional requirement $B > A^2$ [1]. In a replica of the smooth regularization trick of refs. [7] or [11] we also have to introduce an analytic continuation of the semiaxis of r to the whole line modified only by a small, local deformation of this integration path $r = r(t)$, $t \in I\!R$ near the origin. In combination with the above complex rotation of one of the couplings this guarantees the overall $\mathcal{PT}$ symmetry of the whole complexification.

In the light of the detailed analysis of the EH problem as available in the preprint [16] the normalizable wave functions remain proportional to Jacobi polynomials and for all the non-negative integers $n \leq n_{max} < A-1$ also the energies preserve their familiar form

$$E_n^{(EH)} = -\left(A-n-1\right)^2 + \frac{\beta^2}{(A-n-1)^2}, \qquad n = 0, 1, \ldots, n_{max}. \tag{5}$$

Still, one encounters many unusual features in this new real spectrum. A deeper inspection reveals the paradox that an *increase* of the repulsion $A \to A+\delta$ *lowers* the energy, etc.

4.3 $\mathcal{J}R$, Pöschl-Teller case

In our last class of singular forces

$$V^{(PT)}(r) = -\frac{\alpha^2 - 1/4}{\cosh^2 r} + \frac{\beta^2 - 1/4)}{\sinh^2 r}$$

let us preserve the real couplings $\alpha > 0$, $\beta > 0$ and employ just the minimal $\mathcal{PT}$ symmetrization $r(t) = t - i\,\varepsilon$ with any $\varepsilon \in (0, \pi/2)$. A significant novelty of this model lies in its behaviour at short distances. To the first order in the small $\varepsilon > 0$ the estimate

$$\frac{1}{\sinh^2(x-i\varepsilon)} = \frac{\sinh^2(x+i\varepsilon)}{(\sinh^2 x + \sin^2\varepsilon)^2} = \frac{1}{\sinh^2 x} + 2i\varepsilon\frac{\cosh x}{\sinh^3 x} + \mathcal{O}(\varepsilon^2) \tag{6}$$

indicates a clear prevalence of the purely imaginary and strongly singular force. This is is one of sources of the wealth of its spectrum. Its detailed properties have been described in preprint [17]. In the $\varepsilon-$independent explicit formula

$$E^{(PT)} = E_n^{(\sigma,\tau)} = -(2n+1+\sigma\alpha+\tau\beta)^2 < 0$$

the construction leaves the pair of free signs $\sigma, \tau = \pm 1$. The maxima of the non-negative principal integers $n = n(\sigma,\tau) \leq n_{max}^{(\sigma,\tau)}$ must be smaller than the quantity $-(\sigma\alpha+\tau\beta+1)/2 < \infty$ so that the whole spectrum is finite. For the appropriate sizes of the couplings it proves composed of (up to) three non-empty parts,

$$\begin{array}{lll} E_n^{(-,-)} < 0, & 0 \leq n \leq n_{max}^{(-,-)}, & \alpha+\beta > 1, \\ E_n^{(-,+)} < 0, & 0 \leq n \leq n_{max}^{(-,+)}, & \alpha > \beta+1, \\ E_n^{(+,-)} < 0, & 0 \leq n \leq n_{max}^{(+,-)}, & \beta > \alpha+1. \end{array} \tag{7}$$

An additional physical boundary condition must have been imposed in the real $\varepsilon \to 0$ limit [18]. This would fix the unique signs $\sigma = -1$ and $\tau = +1$. In contrast, all the complexified potentials with $\varepsilon \neq 0$ are regular, $|V^{(RPT)}(x)| < const < \infty$. No additional constraint is needed and the spectrum is much richer.

References

[1] F. Cooper, A. Khare and U. Sukhatme, Phys. Rep. 251 (1995) 267.

[2] C. M. Bender and S. Boettcher, Phys. Rev. Lett. 24 (1998) 5243.

[3] E. Delabaere and F. Pham, Phys. Lett. A 250 (1998) 25 and 29;
C. M. Bender, S. Boettcher and P. N. Meisinger, J. Math. Phys. 40 (1999) 2201.

[4] C. M. Bender and K. A. Milton, Phys. Rev. D 55 (1997) R3255 and D 57 (1998) 3595 and J. Phys. A: Math. Gen. 32 (1999) L87.

[5] E. Caliceti, S. Graffi and M. Maioli, Commun. Math. Phys. 75 (1980) 51.

[6] G. Alvarez, J. Phys. A: Math. Gen. 27 (1995) 4589;
F. Fernández, R. Guardiola, J. Ros and M. Znojil, J. Phys. A: Math. Gen. 31 (1998) 10105.

[7] V. Buslaev and V. Grecchi, J. Phys. A: Math. Gen. 26 (1993) 5541.

[8] C. M. Bender and S. Boettcher, J. Phys. A: Math. Gen. 31 (1998) L273;
M. Znojil, J. Phys. A: Math. Gen. 32 (1999) 7419;
M. Znojil, submitted and arXiv: math-ph/0002036v2.

[9] M. Znojil, Phys. Lett. A. 264 (1999) 108.

[10] F. Cannata, G. Junker and J. Trost, Phys. Lett. A 246 (1998) 219;
C. M. Bender, S. Boettcher, H.F. Jones and Van M. Savage, J. Phys. A: Math. Gen. 32 (1999) 6771.

[11] M. Znojil, Phys. Lett. A 259 (1999) 220.

[12] R. G. Newton, Scattering Theory of Waves and Particles (Springer Verlag, New York, 1982).

[13] G. Levai and M. Znojil, submitted and arXiv: quant-ph/0003081.

[14] B. Bagchi and R. Roychoudhury, J. Phys. A: Math. Gen. 33 (2000) L1.

[15] M. Znojil, J. Phys. A: Math. Gen. 33 (2000) L61.

[16] M. Znojil, arXiv: quant-ph/9912027 and math-ph/0002017, unpublished.

[17] M. Znojil, arXiv: quant-ph/9912079, unpublished.

[18] G. Lévai, J. Phys. A: Math. Gen. 22 (1989) 689;
M. Znojil, arXiv: quant-ph/9811088, to appear in Phys. Rev. A.

Acknowledgement

Partially supported by the grant Nr. A 1048004 of the Grant Agency of the Academy of Sciences of the Czech Republic.