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A generalized Morse asymmetric potential and multiplets of its non-numerical exact bound states

Miloslav Znojil
Dept. Theor. Nucl. Physics, Institute of Nuclear Physics ASCR,
250 68 Řež, Czech Republic
e-mail: znojil@ujf.cas.cz

Abstract

The solvable Morse model with its asymmetric phenomenological potential $V(r) = A\ \{1-\exp[-\mu\,(r-r_\alpha)]\}^2$ is generalized: Via a computer-assisted algebraic construction, we show that for certain three-component phenomenological potentials $V(r) = A\ \{1-\exp[-\mu\,(r-r_\alpha)]\}^2 + B\ \{1-\exp[-\mu\,(r-r_\beta)]\}^3 + C\ \{1-\exp[-\mu\,(r-r_\gamma)]\}^4$ the closed and exact bound states may exist not only in singlets, but also in doublets and triplets.

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1 Introduction and summary

The one-dimensional Schrödinger equation

$$[-\frac{d^2}{dr^2} + V(r)]\,\psi(r) = E\,\psi(r), \qquad r \in (-\infty, \infty). \tag{1}$$

with the so called Morse potential

$$V(r) = A\{1 - \exp[-\mu(r - r_\alpha)]\}^2, \quad A > 0, \quad \mu > 0 \tag{2}$$

is well known and exactly solvable in terms of Laguerre polynomials. The specific shape of potential (2) also explains its immediate phenomenological appeal in molecular physics and quantum chemistry [1].

In the latter setting a "more realistic" re-interpretation of our equation (1) is usually preferred, with a three-dimensional, s-wave choice of the range of coordinates $r \in (0, \infty)$. During this transition to an unsolvable equation, the related errors are usually made negligible by an *ad hoc* assumption of a very large value of the parameter $r_\alpha \gg 1$ [2].

In the contemporary molecular phenomenology, the solvable (or, in three dimensions, "almost solvable") Morse model represents just a highly schematic zero-order approximation (say, to the – often *ab initio* – calculated energy surfaces [3]). Hence, naturally, one may try to introduce corrections and contemplate the next, three-term interaction

$$V(r) = A\,\left(1 - \mathrm{e}^{-\mu\,(r - r_\alpha)}\right)^2 + B\,\left(1 - \mathrm{e}^{-\mu\,(r - r_\beta)}\right)^3 + C\,\left(1 - \mathrm{e}^{-\mu\,(r - r_\gamma)}\right)^4 \tag{3}$$

with $C > 0$ for, say, a more reliable prediction of a vibrational molecular spectrum [4].

Methodically, the simplicity and seemingly “next-to-solvable” character makes our generalized potential (3) even more challenging. One might study its relationship to the very similar Morse-like interaction terms which emerge in the so called Toda systems on a lattice [5]. In particular, our Schrödinger-like equation may appear during the analysis of stability of their solitons [6] or, in a way proposed by one of the referees, one might immediately identify our eq. (1) + (3) and some linear differential equations which emerge in the related inverse-scattering formalism (ref. [7], p. 629).

In accord with Turbiner [8], another independent motivation for our study may stem from the possible existence of certain underlying, “quasidynamical” Lie algebras. Whenever they exist, these algebras (plus the standard theory of their representations) offer a simple explanation of existence of a broad class of partially solvable (so called quasi-exactly solvable, QES) systems (cf., e.g., our recent review [9] for more details).

In what follows, we shall see that our eq. (1) + (3) may easily be identified with one of the known QES systems. Unfortunately, its Lie-algebraic treatment will be shown to offer just a single isolated QES eigenvalue. From the point of view of several important applications (the Hill-determinant [10] and perturbative [11] methods, to name just a few), this is not enough. In the present paper we intend to show that in our Schrödinger bound state problem (1) the exact one-dimensional solvability of the Morse model may still be preserved for forces of the more flexible form (3) and, let us re-emphasize, **at several different energy levels**.

A priori, the possible existence of the multiple QES solutions was not clear. Empirically, we even succeeded in showing that, as a rule, the simplest possibilities regularly fail to reveal any multiplets at all. During our research, such a phenomenon proved extremely discouraging. Only a strong (mostly phenomenological and quantum-chemical [2, 7]) practical relevance of the generalized Morse potentials as well as a theoretical importance and possible algebraic background of the existence of multiplets [9] kept us moving to the more complicated situations and, after all, to the affirmative answers.

Our presentation of results will start by a concise introduction into the general QES problem (section 2). In detail, we shall study the suitable choice of variables (subsection 2.1) and the structure of the underlying (namely, coupled and nonlinear) "simultaneous solvability" algebraic equations (subsection 2.2). Next, we shall find and describe some of their explicit solutions which form the elementary bound-state doublets (section 3). For clarity, we shall distinguish between the trivial (subsection 3.1) and next-to-trivial (subsection 3.2) cases. Similarly, in section 4, we shall demonstrate the existence of triplets.

The contemporary progress in the symbolic manipulations on computers [12] was of significant importance and crucial help. With their assistance, the search for solutions proved succesful at last, in spite of the unpleasant fact of life that there are no obvious theorems which would guarantee the simultaneous existence of several bound states "globally", in a way paralleling some other QES examples [13, 14].

Our results were obtained via an extensive use of computer algebra. Our first seven triplet solutions (namely, the four roots in subsection 4.1 and the three results of subsection 4.2) involve the "elementary" doublets of subsection 3.1. Together with the first "less elementary" closed solution obtained in subsection 4.3, all the first eight constructions remain fully non-numerical. The other possible (and, presumably, not non-numerical) triplet solutions are also sampled here as our last, ninth solution in subsection 4.3.

Our construction demonstrates a usefulness of the generalization (3) of Morse forces. The complexity of its multiplets is not prohibitive – up to a single exception, all our closed triplet solutions (see below) were purely non-numerical. The elementary character of these triplets resulted from the encountered (and unexpected) numerous simplifications in the underlying algebra. An analysis of the deeper reasons for this (as well as of some further open questions, e.g., a conjectured absence of the "neighbouring" doublets, etc.) seems to be of a less immediate physical appeal and interest. It might deserve a more mathematically oriented study in the future.

2 Elementary bound states

2.1 The Liouvillean change of variables

Let us first re-scale the coordinate $r \to \rho r$ in such a way that $\mu = 2$ in our one-dimensional potentials. The change of variables

$$r \to R = R(r) \equiv \exp(-r) \in (0, \infty), \qquad \psi(r) \to \phi(R) = R^{1/2}\psi(r) \tag{4}$$

may then be performed in accord with Liouville [15]. This transforms eq. (1) into another, equivalent radial Schrödinger equation

$$[-\frac{d^2}{dR^2} + \frac{L(L+1)}{R^2} + V^{(Liouv.)}(R)]\ \phi(R) = \varepsilon\ \phi(R) \tag{5}$$

with the formal L-th wave potential

$$\begin{aligned} V^{(Liouv.)}(R) &= g_2\, R^2 + g_4\, R^4 + g_6\, R^6, \\ g_2 &= A \exp(4r_\alpha) + 3B \exp(4r_\beta) + 6C \exp(4r_\gamma), \\ g_4 &= -\,B \exp(6r_\beta) - 4C \exp(6r_\gamma), \\ g_6 &= C \exp(8r_\gamma) > 0. \end{aligned} \tag{6}$$

We see that our generalized Morse model must share at least some of the distinctive features of the QES sextic oscillators. Even the ordinary Morse potential (with $B = C = 0$) may be understood as equivalent to the exactly and completely solvable harmonic oscillator itself in this language.

In the "quasi-realistic" s-wave problem (with the approximative half-axis domain, not to be studied in any detail in what follows), our new, "Liouvillean" system and forces (6) become confined to a box with infintely high walls at a finite radius, $R \in (0, 1)$. Under certain circumstances, this might represent an important numerical merit [16]. Here, we intend to proceed purely non-numerically.

One of the most attractive properties of sextic oscillators (6) lies in their QES property. The corresponding multiplets of the exact solutions of eq. (5) + (6) (with an elementary harmonic-oscillator-like structure of wavefunctions) were originally derived, without any use of Lie algebras, in a detailed paper by Singh et al. [13]. This publication was preceded by a virtually unnoticed discovery of this possibility by Hautot [17], cca. ten years sooner. Both these studies, not too often cited, inspired a boom of similar constructions during the last fifteen years (cf., e.g., their short review [9] as well as a randomly chosen sample [18] of their re-current re-discoveries).

In contrast to the QES sextic example, the Liouvillean change of variables will assign each element of the Singh's multiplet of wavefunctions to a different generalized Morse potential (3). The explanation is easy. Transformation (4) replaces the original energy E by a new parameter of an "angular momentum",

$$L = -\tfrac{1}{2} + \sqrt{A+B+C-E} > -\tfrac{1}{2}. \tag{7}$$

Thus, we may write

$$E = A + B + C - Z^2, \qquad Z = L + \tfrac{1}{2} > 0. \tag{8}$$

Vice versa, the "new energy" ε abbreviates

$$\varepsilon = 2A \exp(2r_\alpha) + 3B \exp(2r_\beta) + 4C \exp(2r_\gamma) \tag{9}$$

and remains fixed by the potential (3). As a consequence, a straightforward translation of the Singh's construction generates a mere set of the so called Sturmians [14] rather than a multiplet of bound states belonging to the same potential.

The feasibility of construction of a genuine multiplet of elementary bound states (so useful, e.g., in perturbation theory [19]) remains an open question in the present generalized-Morse setting. This inspired our paper which is going to pay attention to the difficult problem of multiplets.

2.2 The algebraic QES conditions

Morse [1] derives his potential via an elementary special-function Ansatz. Here, a more general result will be achieved via a polynomial Ansatz

$$\phi^{(elementary)}(R) = e^{-G(R)}\ R^{L+1} \sum_{n=0}^{N} h_n R^{2n}, \qquad L > -\tfrac{1}{2}. \tag{10}$$

Without any further specification of its coefficients, with an arbitary integer $N < \infty$ and with a WKB-compatible exponent

$$G(R) = \frac{1}{4}\lambda R^4 + \frac{1}{2}\xi R^2, \qquad \lambda = \sqrt{C}\exp(4r_\gamma) > 0, \qquad \xi = g_4/(2\lambda) \tag{11}$$

our normalizable (i.e., physical and correct) bound state Ansatz transforms differential equation (1) with the generalized Morse potential into its algebraic realization,

$$\begin{gathered}
B_n h_{n+1} = C_n^{(0)} h_n + C_n^{(1)} h_{n-1}, \\
B_n = 4\,(n+1)\,(n+Z+1), \qquad Z \equiv L + \tfrac{1}{2} \\
C_n^{(0)} = 2\xi(2n+Z+1) - \varepsilon \\
C_n^{(1)} = 2\lambda(2n+Z) - \xi^2 + g_2 \\
n = 0, 1, \ldots \\
h_0 \neq 0, \quad h_N \neq 0, \quad h_{N+1} = h_{N+2} = \ldots = 0.
\end{gathered} \tag{12}$$

At an arbitrary energy (either the "new" ε or "old" E), the latter form of Schrödinger equation happens to be exactly solvable,

$$h_{n+1} = \frac{h_0}{\prod_{k=1}^{n} B_k} \det H^{[n]}, \quad n = 0, 1, \ldots \tag{13}$$

$$
H^{[n]} = \begin{pmatrix} C_0^{(0)} & -B_0 & & & & \\ C_1^{(1)} & C_1^{(0)} & -B_1 & & & \\ 0 & C_2^{(1)} & C_2^{(0)} & -B_2 & & \\ & & & \cdots & & \\ & \cdots & 0 & C_n^{(1)} & C_n^{(0)} \end{pmatrix} .
$$

The existence of our closed bound-state solutions proves formally equivalent to the validity of the only two remaining termination or "selfconsistency" conditions

$$
C_{N+1}^{(1)} \; (= 2\lambda(2N + Z + 2) - \xi^2 + g_2) = 0 \tag{14}
$$

(cf. $h_{N+1} = h_{N+2} = 0$ and $h_N \neq 0$) and

$$
\det H^{[N]} = 0 \tag{15}
$$

(the requirement $h_{N+1} = 0$ in the language of the determinantal eq. (13)).

After a short inspection of the former constraint (14) we notice that whenever we fix all the couplings in our generalized Morse potential (3), the only possible source of variation of the "new angular momentum" parameters L or Z is just the dimension N itself,

$$
Z = Z(N) = (g_4)^2/(2\lambda)^3 - g_2/(2\lambda) - 2N - 2 > 0, \qquad N = 0, 1, \ldots \,. \tag{16}
$$

In an ascending order, all the "solvable" energy levels (if any) become numerated by this N in accord with eq. (8),

$$
\begin{aligned}
E_N^{(solvable)} &= -4\,N^2 + 4Nz + A + B + C - z^2, \\
N = 0, 1, \ldots, N_{max}, &\qquad N_{max} = -\, entier[-\,z/2], \\
z \; (\equiv Z(0)) &= (g_4)^2/(2\lambda)^3 - g_2/(2\lambda) - 2.
\end{aligned} \tag{17}
$$

Vice versa, at a fixed N, equation (16) enables us to eliminate, say, g_2 as a function of $z = Z(0)$. This reduces the whole set of five free constants in Schrödinger equation (5) to a mere quadruplet.

A re-scaling of R's enables us to assign any numerical value to r_γ, g_6 or λ. This means that on the r-line, the $r \to -r$ asymmetry of our forces $V(r)$ leaves the position of the origin virtually undetermined. After we pick up $\lambda = 1/2$ for definitness, we are left with three independent free parameters, say, $x\ (= \varepsilon)$, $y\ (= -2\,g_4\ \equiv -\,g_4/\lambda)$ and $z\ (= Z(0) > 2\,N \geq 0)$.

The proof of existence of at least one elementary bound state energy level has to be based now on an analysis of our last constraint (15). One of the available three free parameters must be eliminated by it. With the relation (14) taken into account, the constraint reads

$$\det Q^{[N]}(x,y,z) = 0. \tag{18}$$

Here, $Q^{[N]}(x,y,z)$ denotes the $N+1$-dimensional tridiagonal matrix

$$\left(\begin{array}{ccccc} x+y(Z+1) & 4+4Z & & & \\ 2N & x+y(Z+3) & 16+8Z & & \\ & \cdots & & & \\ & 4 & x+y(Z+2N-1) & 4N^2+4NZ & \\ & & 2 & x+y(Z+2N+1) \end{array} \right) \tag{19}$$

i.e., matrix $H^{[N]}$ with $Z = Z(N) = z - 2N > 0$. Mathematically speaking, we now only have to show that the roots of eq. (18) are real. This is not difficult: Due to the positivity of all the off-diagonal matrix elements in eq. (19), the tridiagonal matrix $Q^{[N]}(x,y,z)$ may easily be symmetrized. Then, algebraic equation (18) may be visualized as a diagonalization of a real and symmetric matrix. Up to a possible degeneracy [20], this always gives an $N+1$-plet of “Sturmian” real eigenvalues x or y.

3 The construction of doublets

In a study of pairs of elementary bound state solutions, the necessary condition of their existence

$$\begin{array}{ll} \det Q^{[N]}(x,y,z)=0, & N\geq 0,\\ \det Q^{[M]}(x,y,z)=0, & M>N \end{array} \tag{20}$$

has to be understood as a coupled pair of polynomial equations (18) in two variables, say, x and y. Their choice from a menu

$$\begin{array}{ll} x+y\,(z+1)=0, & N=0,\\ x^2+2\,y\,z\,x+y^2\,(z^2-1)+8\,(1-z)=0, & N,M=1,\\ x^3+y\,x^2\,(3\,z-3)+x\,[y^2\,(3\,z^2-6\,z-1)+16\,(5-2\,z)] & \\ +y\,(z-3)\,[y^2\,(z^2-1)+16\,(1-2\,z)]=0, & N,M=2,\\ x^4+y\,x^3\,(4\,z-8)+x^2\,[y^2\,(6\,z^2-24\,z+14)+80\,(4-z)] & \\ +4\,y\,x\,[y^2\,(z^3-6\,z^2+7\,z+2)-8\,(5\,z^2-30\,z+34)] & \\ +(z-5)\,[y^4\,(z-3)\,(z^2-1)-16\,y^2\,(5\,z^2-15\,z+4) & \\ +576\,(z-3)]=0, & N,M=3,\\ \ldots & \end{array} \tag{21}$$

is instructive in showing the mathematical difficulty of the problem: Doublets will be defined as a solution of a coupled pair (20) of nonlinear algebraic equations (21).

3.1 The simplest choice, $N = 0$

At $N = 0$, the linear item in (21) leads to the unique and real root $x = -y(z+1)$. The positivity of $Z(M) = z - 2M > 0$ is to be demanded as equivalent to the correct $r \to \infty$ asymptotics (and finite norm) of the wavefunctions. A priori, the simplest doublets of elementary bound states might then result from an insertion of x $(= \varepsilon(0))$ in the rest of our menu. We get the new sequence of restrictions

$$\begin{array}{c}
8\,(1-z) = 0, \qquad M = 1, \qquad z > 2, \\
64\,y\,(z-2) = 0, \qquad M = 2, \quad z > 4, \\
576\,(3-z)[y^2 - (z-5)] = 0, \qquad M = 3, \qquad z > 6, \\
2048\,y\,(z-4)\,[3\,y^2 - 2\,(4\,z - 25)] = 0, \qquad M = 4, \qquad z > 8, \\
12800\,(5-z)\,[6\,y^4 + y^2\,(217 - 29\,z) + 9\,(z-9)\,(z-7)] = 0, \\
M = 5, \qquad z > 10, \\
221184\,y\,(z-6)\,[5\,y^4 + y^2\,(322 - 37\,z) \\
+4\,(8\,z^2 - 148\,z + 675)] = 0, \qquad M = 6, \qquad z > 12, \\
\ldots
\end{array} \tag{22}$$

imposed upon y and z.

The simplest choice of degrees $(N, M) = (0, 1)$ implies that $Z(M) = -1$ remains negative and, hence, unacceptable. The related $M = 1$ wavefunction (10) does not behave properly at $R = 0$, i.e., at $r \gg 1$. The choice of larger M's is more rewarding and leads to the following abundance of the real and acceptable roots $y = y_m(M, z)$,

$$
\begin{aligned}
y_1(2,z) &= 0,\\
y_{1,2}(3,z) &= \pm\sqrt{z-5},\\
3\,y_{1,2}(4,z) &= \pm\sqrt{24\,z-150},\\
6\,y_{1,2,3,4}(5,z) &= \pm\sqrt{87\,z-651\ \pm 3\,\sqrt{625\,z^2-9130\,z+33481}},\\
y_{1,2,3,4}(6,z) &= \pm\sqrt{\pm\sqrt{729\,z^2-11988\,z+49684}+37\,z-322}/\sqrt{10},\\
&\ldots\,.
\end{aligned}
\tag{23}
$$

All these doublet roots stay real under the automatically satisfied conditions $z > c(M)$ with $c(3) = 5,\ c(4) = 6.25,\ c(5) = 9,\ c(6) = 10.34$ etc.

3.2 The next case, $N = 1$

At $N = 1$, the quadratic polynomial (21) possesses a pair of real roots

$$
\varepsilon(1) = x_{1,2} = -y\,z \pm \sqrt{y^2+8\,z-8}
\tag{24}
$$

which define the two different potentials (3). Each of the roots remains real in a non-empty domain $8\,z + y^2 \geq 8$ $(z \geq 1)$ which incorporates all the physical z's again, $Z(M) = z - 2M \in (1 - 2\,M, \infty)$.

The elimination of $\varepsilon(1)$ generates the new hierarchy of equations

$$
\begin{aligned}
8\,(3-z)\,(3\,\sqrt{y^2+8\,z-8}-y) &= 0, \qquad M=2,\\
128\,(z-4)\,(2\,y\,\sqrt{y^2+8\,z-8}-y^2-12) &= 0, \qquad M=3,\\
&\ldots\,.
\end{aligned}
\tag{25}
$$

An unacceptable imaginary solution emerges at $M = 2$, but the further steps of the search with $M \geq 3$ are succesfull and yield the physical parameters

$y^2 = y_j^2(M,z)$ as real roots of polynomials of degree $M-1$,

$$\begin{aligned} \sqrt{3}\,y_{1,2}(3,z) &= \pm 2\sqrt{2\sqrt{4\,z^2-14\,z+19}-4\,z+7} \\ &= \pm 2\sqrt{2\sqrt{4\,a^2+34\,a+79}-4\,a-17}, \qquad a = z-6 > 0, \\ &\ldots\,. \end{aligned} \tag{26}$$

4 The triplets of elementary bound states

Both equations (22) and (25) and their elementary-doublet solutions (23) and (26) depend on the last free parameter z. We may try to fix its value by an additional contraint,

$$\begin{aligned} \det Q^{[N]}(x,y,z) &= 0, \qquad N \geq 0, \\ \det Q^{[M]}(x,y,z) &= 0, \qquad M > N, \\ \det Q^{[K]}(x,y,z) &= 0, \qquad K > M \end{aligned} \tag{27}$$

in combination with the appropriate boundary-condition formula

$$Z(K) = z - 2K > 0. \tag{28}$$

This would guarantee the existence of triplets of elementary bound-state wavefunctions (10) in the given generalized Morse potential (3).

In the light of our preceding results, the absence of doublets implies the absence of triplets at any $(N,M,K) = (0,1,K)$ and $(N,M,K) = (1,2,K)$ (and, we feel tempted to conjecture, at any $(N,M,K) = (N,N+1,K)$). We may also cross out $(N,M,2\,k+1) = (0,2,2\,k+1)$ with $k = 1,2,\ldots$ since all the related solutions read $z = z_i(N,M,K) = z_i(0,2,2\,k+1) = 2k+2i+1$, $i = 0,1,\ldots,k$ and contradict eq. (28). Finally, in the case of $(N,M,K) = (0,2,2\,t)$ with $t = 2,3,\ldots$, with "quasi-trivial" solutions

$x = y = 0$ and with *arbitrary* z's, it is easy to demonstrate that all the related potentials (3) degenerate back to the ordinary Morse force after a re-scaling $\mu \to \mu/2$. Within our present formalism, the quasi-trivial equations (12) with disappearing h_{2k+1}'s degenerate to recurrences for $q_k \equiv h_{2k}$'s,

$$q_1 = -4tq_0/(16 + 8\,Z), \quad q_2 = -(4t-4)q_1/(64 + 16\,Z), \quad \ldots\,. \tag{29}$$

These equations are solvable, and their solution

$$q_k\ (\equiv h_{2k}) = (-1)^k 4^{-k} \left(\begin{array}{c} t \\ k \end{array} \right) \frac{\Gamma(1 + Z/2)}{\Gamma(k + 1 + Z/2)} q_0, \quad k = 0, 1, \ldots, t \tag{30}$$

may easily be recognized as a definition of Laguerre polynomials. In this way, the consistency of our approach becomes confirmed *a posteriori*.

At the next few nontrivial (N, M, K)'s, let us now perform a (computer-assisted) search for nontrivial triplets of the terminating bound states (10).

4.1 A systematic study of $(N, M, K) = (0, 3, K)$

In the light of eq. (28), all the common roots $z_0(N, 3, K) = 3$ and their K-dependent partners $z_1(0, 3, K) = K + 3$ may be ignored as unphysical. An inspection of eq. (22) and an elimination of $y^2 = z - 5$ via its third row shows that $K = 4$ and $K = 6$ offer no other real roots, while $K = 5$ gives just a too small (="unphysical") $z_2 = 23/7 \approx 3.29 \ll 10$. The first success is only encountered at $K = 7$. Via a quadratic equation and besides an unphysical $z_2 \approx 3.66$, we get our first physical solution

$$z^{\{1\}} = z_3(0, 3, 7) = \left(126 + \sqrt{7417}\right)/11 \approx 19.28381877\,(\,> 14). \tag{31}$$

Similarly, $K = 8$ gives the unphysical $z_2 \approx -4.72$ and our second acceptable solution

$$z^{\{2\}} = z_3(0,3,8) = \left(563 \ + 18\sqrt{3039}\right)/91 \approx 17.09107983 \ (> 16). \qquad (32)$$

In the same manner, the forthcoming sequence of equations

$$\begin{array}{c} 81\,z^2 - 1694\,z + 5325 = 0, \qquad K = 9, \\ 374\,z^3 - 20441\,z^2 + 229555\,z + 245840 = 0, \qquad K = 10, \\ 1729\,z^4 - 55730\,z^3 - 470604\,z^2 + 20580178\,z - 71914325 = 0, \quad K = 11, \\ \ldots \end{array} \qquad (33)$$

has to be solved step-by-step. It leads to the unphysical $K = 9$ pair of roots $z_{2,3} = \left(847 \ \pm 2\sqrt{71521}\right)/81 \approx \{17.06, 3.85\}$ followed by the three $K = 10$ solutions, etc. In the latter case the unphysical pair $z_{2,3} \approx \{17.55, -0.98\}$ is accompanied by our third physical root

$$\begin{array}{l} z^{\{3\}} = z_4(0,3,10) = \sqrt{160273771}\,\sin[(\theta+\pi)/3]/561 + 20441/1122 \\ \quad \approx 38.0864 > 20, \qquad \mathrm{tg}\theta = 8\,\frac{\sqrt{72627985461657814269157509940304434 7355}}{24271124673483503555 85}. \end{array} \qquad (34)$$

This is to be complemented by the four real roots $z_{2,3,4,5}$ at $K = 11$. Besides their unphysical subset $z_{2,3,4} \approx \{-19.35,\ 4.02,\ 18.25\}$, our fourth well-behaved solution is born,

$$z^{\{4\}} = z_5(0,3,11) = \ldots \approx 29.319614 > 22. \qquad (35)$$

In principle, this is still a non-numerical result [20], but the study of the higher K's already seems to require an alternative, purely numerical approach to the corresponding polynomial equations (33).

4.2 The case of $(N, M, K) = (0, 4, K)$

The common root $z_0(N, 4, K) = 4$, the less trivial and K-dependent integer $z_1(0, 4, K) = K + 4 < 2\,K$, the fractional and unique solutions of linear equations $z_2(0, 4, 5) = 461/77 \approx 5.99$, $z_2(0, 4, 6) = 115/28 \approx 4.11$ and $z_2(0, 4, 8) = 1433/320 \approx 4.48$ at $K = 5,\ 6$ and $K = 8$ as well as, finally, the two $K = 7$ roots $z_{2,3}(0, 4, 7) = (197\ \pm 6\sqrt{33829})/221 \approx \{5.88,\ 4.10\}$ of the underlying quadratic equation may be ignored. All of them violate the normalizability condition (28). Only the subsequent $M = 4$ counterparts of eq. (33), namely,

$$
\begin{array}{c}
33649\, z^3 - 1432887\, z^2 + 8607483\, z - 7813045 = 0, \qquad K = 9, \\
16016\, z^3 - 399768\, z^2 - 1531515\, z + 14532070 = 0, \qquad K = 10, \\
96135\, z^4 + 45620\, z^3 - 87827630\, z^2 + 696931300\, z - 1172415169 = 0, \\
K = 11, \\
\ldots
\end{array}
\tag{36}
$$

become productive. At $K = 9$ they give the (still non-numerical) unphysical solutions $z_2(0, 4, 9) \approx 5.90$, $z_3(0, 4, 9) \approx 1.11$ plus our fifth satifactory root

$$
\begin{array}{c}
z^{\{5\}} = z_2(0, 4, 9) \\
= 8\sqrt{8224066447}\,\sin[(\phi + \pi)/3]/33649 + 477629/33649 \approx 35.57662, \\
\mathrm{tg}\phi = 5\,\frac{\sqrt{16431691081236520032956144866990550350030008502}}{26064026995931543206387 8}.
\end{array}
\tag{37}
$$

Similarly, the unphysical pair $z_2(0,4,10) \approx -7.03$ and $z_3(0,4,10) \approx 4.74$ is complemented by the physical

$$\begin{gathered} z^{\{6\}} = z_4(0,4,10) \\ = \sqrt{1620838101}\,\sin[(\psi+\pi)/3]/2002 + 16657/2002 \approx 27.24785, \\ \mathrm{tg}\psi = 26217\,\frac{\sqrt{174472147881820414117911486288006835 5}}{581545435433055710917 30}. \end{gathered} \tag{38}$$

The list of non-numerical possibilities seems completed by a biquadratic equation (36) at $K = 11$. The inacceptable triplet of its too small roots $z_2(0,4,11) \approx -33.97$, $z_3(0,4,11) \approx 2.41$, $z_4(0,4,11) \approx 5.93$ is accompanied by our last $N = 0$ acceptable solution

$$z^{\{7\}} = z_5(0,4,11) = \ldots \approx 25.153897 > 22\ldots.. \tag{39}$$

Again, the latter root is non-numerical [20] but its explicit non-numerical form proves impractical for virtually any purpose.

4.3 The more complicated series $(N,M,K) = (1,3,K)$

At $N = 1$, we have to expand the even powers of the square root $v = \sqrt{64\,(4\,z^2 - 14\,z + 19)/9}$ in terms of z's. This is a clumsy but straightforward procedure – we get equations which are linear in v and of the $(K-1)$-st order in the second variable z,

$$\begin{gathered} 7075\,z^3 - 15\,z^2\,(80\,v + 5739) + 15\,z\,(796\,v + 19651) \\ -24780\,v - 271915 = 0, \quad K = 4, \\ \ldots\,. \end{gathered} \tag{40}$$

The elimination of v and its squaring leads to another series of equations

$$\begin{gathered}13475\, z^4 - 196000\, z^3 + 725114\, z^2 - 659040\, z - 272925 = 0, \qquad K = 4, \\ 560\, z^3 - 8351\, z^2 + 33096\, z - 33260 = 0, \qquad K = 5, \\ \ldots . \end{gathered} \tag{41}$$

Their K-dependence ceases to be trivial – we get a polynomial of the eighth degree at $K = 6$, etc. We may omit the redundant details now. Besides the unphysical $z_2(1,3,4) = -0.304$, $z_3(1,3,4) = 15/7 \approx 2.14$ and $z_4(1,3,4) \approx 3.31$, we get our final non-numerical physical $K = 4$ solution

$$z^{\{8\}} = z_2(1,3,4) = \ldots \approx 9.3974688 > 8. \tag{42}$$

Indeed, the next equation produces just inacceptable roots $z_2(1,3,5) = 9.18$, $z_3(1,3,5) = 4.19$, $z_4(1,3,5) = 1.54$ while, as already mentioned, the subsequent $K = 6$ item in eq. (41) would already require a purely numerical treatment.

By the way, the specialized numerical algorithms prove extremely efficient in practice. Their careful use even enables us to prove the existence of the physical solutions of the type

$$z^{\{9\}} = z_2(1,3,6) = \ldots \approx 15.07882 > 12 \tag{43}$$

i.e., roots and bound states without a consequently non-numerical character.

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