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NONLINEARIZED PERTURBATION THEORIES

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Abstract

A brief review is presented of the two recent perturbation algorithms. Their common idea lies in a not quite usual treatment of *linear* Schrödinger equations via *nonlinear* mathematical means.

The first approach (let us call it a quasi-exact perturbation theory, QEPT) tries to get the very zero-order approximants already "almost exact", at a cost of leaving the higher-order computations more complicated. Technically, it constructs and employs solutions of certain auxiliary nonlinear systems of algebraic equations for the suitable zero-order couplings and energies.

The second approach (a fixed-point perturbation theory, FPPT) pays more attention to the hogher-order corrections. Its purpose lies in an improvement of construction of unperturbed propagators or, alternatively, of the closely related (so called effective) finite-dimensional auxiliary Hamiltonians. On a technical level, it employs a factorization interpreted via certain nonlinear mappings and, finally, approximates some matrix elements by fixed points of these mappings.

In a broad context of the "generalized Rayleigh-Schrödinger" perturbation strategy, both the prescriptions need just more summations over "intermediate states". QEPT defines its nondiagonal unperturbed propagators in terms of infinite continued fractions. FPPT introduces a further simplification via another finite system of nonlinear algebraic equations for fixed points. Thus, both the subsequent QE and FP steps of

construction share the same mathematics.

1 The two modified perturbation theories

1.1 Motivation

In the various perturbative approaches to particular equations, a key role is often played by computerized algebraic manipulations. In this setting, the present review summarizes and slightly extends my own results contained in papers [1] and [2]. The first reference deals mostly with the nonlinear Magyari equations and QEPT H_0's. The second pair pays attention to the fixed points of nonlinear mappings and to their explicit constructions and use in FPPT. My overall message is a little bit nonstandard: Usually, nonlinear equations (generated, e.g., by phenomenological considerations) are being solved by their formal (e.g., approximative or inverse-scattering) linearization. Here, I am going to sample an efficiency of the opposite strategy and treatment of a linear physical *system* via nonlinear mathematical *techniques*.

In the past, my effort was motivated by immanent limitations of the textbook perturbation theory: Literature abounds with the examples (e.g., the famous quartic anharmonic oscillator is well known to lead to the divergent Rayleigh-Schrödinger series for energies, $E(g) = E^{(0)} + gE^{(1)} + g^2E^{(2)} + \ldots$, at an *arbitrary* value of coupling g). In a less perturbative and/or nonperturbative context, parallel applications of nonlinearizations are also not rare in the literature. One might quote the so called Hill determinant method which defines physical energies as roots of certain infinite-dimensional determinats and leads to a brand new field of applicability of the idea of symmetry (e.g., a hidden symmetry between the physical and spurious roots [3]).

Similar nonlinearizations and spuriosities which may be related to symmetries arise also in the so called Riccati-Padé method of quantum chemistry (cf. ref. [4], with a puzzling threshold – asympotics symmetry), in the universal coupled cluster approach to the general manybody problem (cf., e.g., a review [5] where certain highly sophisticated symmetry between the classical and quantum descriptions has been mentioned) etc.

For the sake of transparency of presentation, we shall start from a particle confined in a particular Flessas - Gallas double well potential [6]

$$V(x) = x^2 + F\frac{g\,x^2}{1+g\,x^2}, \qquad F = F(g) = -4 - 6\,g.$$

Traditionally, in the corresponding Schrödinger equation

$$\left[H^{(HO)} - \frac{F}{1+g\,r^2}\right]\Psi(r) = E\,\Psi(r), \quad E = k^2 - F$$

one perturbes harmonic oscillator $H^{(HO)} = \sum_0^\infty |n\rangle\varepsilon_n\langle n|$, $\varepsilon_n = 4n + 2\ell + 3$, in a way which exhibits an unpleasant quasi-nonperturbative g-dependence ($E(g)$ proves to be strongly curved near the zero-order harmonic oscillators $g \approx 0$).

In our first, nontraditional QEPT approach, we employ and extend the ideas of Lanczos and Whitehead et al [7] and make the eigenvalue problem tridiagonal,

$$\left(\begin{array}{cccc} A_0 & B_0 & & \\ C_1 & A_1 & B_1 & \\ & C_2 & A_2 & B_2 \\ & & \ldots & \end{array}\right)\left(\begin{array}{c} (\varepsilon_0 - E)\,\Psi_0 \\ (\varepsilon_1 - E)\,\Psi_1 \\ (\varepsilon_2 - E)\,\Psi_2 \\ \ldots \end{array}\right) = 0$$

$$A_k = \frac{g}{2}\varepsilon_k + 1 - \frac{F}{\varepsilon_k - E}, \quad B_k = C_{k+1} = g\sqrt{(k+1)(k+\ell+3/2)}.$$

In such a setting, one may generate certain particular, closed-form states even at some nonzero $g = g(N)$'s (cf. [6]).

In the vicinity of these "quasi-harmonic" zero-order solutions, the energies $E(g)$ form the nice and smooth (virtually linear!) lines. This is an intuitive explanation of the high precision of the zero-order QEPT approximants as well as of the marvelous convergence of the related simplified FPPT series near the Gallas states [8].

A characteristic feature of our example is an extreme simplicity of its technical aspects. Indeed, the Magyari-like construction of the QEPT zero order states is virtually trivial [7]. Due to the above mentioned tridiagonality of H_o, one also encounters the easily solvable fixed point equations as well as a closed form of the higher-order fixed-point corrections. Hence, a real technical challenge will only be encountered at the more complicated interactions. A generalized Morse potential will be picked up as an illustration.

1.2 The zero-order QEPT approximations

The Gallas' terminating harmonic-oscillator-like zero-order solutions cannot exist at $\varepsilon_k - E \neq 0$ since $B_k \neq 0$. Thus, one must fix $E = \varepsilon_N$. As a consequence, we may write $\Psi(r) \equiv (1 + g\,r^2) \times \varphi(r)$ and use the $(N+1)$-st row of our tridiagonal algebraic Schrödinger equation as a simple definition of energy. As long as the N-th row then implies that $\varphi^N = 0$ while oscillation theorem adds $\varphi^{N+1} = 0$, $\varphi^{N+2} = 0$, $\ldots$, we arrive, in accord with the above-quoted Whitehead's paper and its slight modification, at the single Magyari equation $Q\varphi = 0$ with an N-dimensional matrix Q,

$$\left(\begin{array}{ccccc} (\varepsilon_0 - \varepsilon_N)A_0 & (\varepsilon_0 - \varepsilon_N)B_0 & & & \\ (\varepsilon_1 - \varepsilon_N)C_1 & (\varepsilon_1 - \varepsilon_N)A_1 & (\varepsilon_1 - \varepsilon_N)B_1 & & \\ & (\varepsilon_2 - \varepsilon_N)C_2 & (\varepsilon_2 - \varepsilon_N)A_2 & (\varepsilon_2 - \varepsilon_N)B_2 & \\ & & \ldots & & \\ & & (\varepsilon_{N-1} - \varepsilon_N)C_{N-1} & (\varepsilon_{N-1} - \varepsilon_N)A_{N-1} & \end{array} \right)$$

and with the subsequent Gallas' observations of the existence, uniqueness and reality of the root $g = g^{(N)}$ or, in the other perspective, of the quasi-oscillator zero-order family of states numbered by the integer N.

1.3 The higher-order FPPT corrections

With the operator $H_0 - E_0 I$ factorized,

$$H_0 - E_0 I = (I + \Omega) \left(\begin{array}{ccccc} F_0 & 0 & \ldots & & \\ 0 & F_1 & 0 & \ldots & \\ 0 & 0 & F_2 & 0 & \ldots \\ & & & \ldots & \end{array} \right) (I + \Omega^+)$$

$$\Omega_{nn} = \Omega_{nn-1} = \ldots = \Omega_{n0} = 0$$

one analyses each zero-order equation

$$H_0 \; |\psi_0\rangle = E_0 \; |\psi_0\rangle$$

quite easily – with $F_0 = 0$ premultiplied by $(I+\Omega)^{-1} = I - \Omega + \Omega^2 - \ldots$, we get

$$Q(I+\Omega^+)|\psi_0\rangle = 0,$$

$$Q = I - |0\rangle\,\langle 0| = \sum_{n=1}^{\infty} |n\rangle\,\langle n|$$

i.e., an equation which becomes immediately solvable,

$$\langle m|\psi_0\rangle = (-1)^m \; \langle 0|\psi_0\rangle \times \det P^{[m]}.$$

$$P^{[m]} = \left(\begin{array}{ccccc} (\Omega^+)_{1,0} & 1 & 0 & \ldots & 0 \\ (\Omega^+)_{2,0} & (\Omega^+)_{2,1} & 1 & 0 & \ldots \\ & & \ldots & & \\ (\Omega^+)_{m-1,0} & \ldots & (\Omega^+)_{m-1,m-3} & (\Omega^+)_{m-1,m-2} & 1 \\ (\Omega^+)_{m,0} & \ldots & (\Omega^+)_{m,m-3} & (\Omega^+)_{m,m-2} & (\Omega^+)_{m,m-1} \end{array} \right).$$

Now, the perturbative higher order formulae remain standard. With $|\tau_0\rangle = 0$ and

$$(H_0 - E_0 I)\; |\psi_k\rangle + (H_k - E_k)\; |\psi_0\rangle + |\tau_{k-1}\rangle = 0, \qquad k = 1,\; 2,\; \ldots\;,$$

$$\left(\begin{array}{cccc} 0 & 0 & \ldots & \\ 0 & F_1 & 0 & \ldots \\ & & \ldots & \end{array} \right) (I+\Omega^+)|\psi_k\rangle + (I+\Omega)^{-1}[(H_k - E_k)|\psi_0\rangle + |\tau_{k-1}\rangle] = 0,$$

$$|\tau_{k-1}\rangle = (H_{k-1} - E_{k-1})|\psi_1\rangle + (H_{k-2} - E_{k-2})|\psi_2\rangle + \ldots (H_1 - E_1)|\psi_{k-1}\rangle$$

we may project out the energies as a mere abbreviation,

$$E_k = \frac{\langle 0|(I+\Omega)^{-1} H_k|\psi_0\rangle + \langle 0|(I+\Omega)^{-1}|\tau_{k-1}\rangle}{\langle 0|(I+\Omega)^{-1}|\psi_0\rangle}$$

and construct the PT expansions in a routine way.

2 FPPT and the factorization of propagators

Technically, a useful nilpotency emerges after an arbitrary approximative truncation of matrices,

$$(\Omega^m)_{n,n+m-1} = (\Omega^m)_{n,n+m-2} = \ldots = (\Omega^m)_{n,0} = 0, \quad m = 1,\ 2,\ \ldots.$$

Renormalizing $|\psi_k\rangle \to |\psi_k\rangle + const\,|\psi_0\rangle$ into $\langle 0|\psi_{k+1}\rangle = 0$, i.e.,

$$\left(\begin{array}{cccc} F_1 & 0 & \ldots & \\ 0 & F_2 & 0 & \ldots \\ & & \ldots & \end{array} \right) (Q + Q\Omega^+ Q) \left(\begin{array}{c} \langle 1|\psi_k\rangle \\ \langle 2|\psi_k\rangle \\ \ldots \end{array} \right) = \left(\begin{array}{c} \langle 1|\rho\rangle \\ \langle 2|\rho\rangle \\ \ldots \end{array} \right)$$

$$|\rho\rangle = -(I+\Omega)^{-1}[(H_k - E_k)|\psi_0\rangle + |\tau_{k-1}\rangle]$$

we get the final formula

$$\langle m+1|\psi_k\rangle =$$

$$= \langle m+1|[Q - Q\Omega^+ Q + (Q\Omega^+ Q)^2 - \ldots + (\pm 1)^m (Q\Omega^+ Q)^m]\times$$

$$\times \left(\begin{array}{cccc} 1/F_1 & 0 & \ldots & \\ 0 & 1/F_2 & 0 & \ldots \\ & & \ldots & \end{array} \right) Q|\rho\rangle, \quad m = 0,\ 1,\ \ldots$$

for wavefunctions. Thus, obviously, the factorization and construction of Ω's is a key technical problem. Its solution may either be based on the use and/or generalization of the so called continued fractions, or on an alternative approximation technique. In the latter context, various FPPT and FPPT-like prescriptions have been proposed and analysed in the related literature (cf., e.g., the list of references in [2]).

The most universal recipe may directly start from the finite submatrices

$$\left(\begin{array}{llll} [R_{n,n} - E_0] & R_{n,n+1} & \ldots & R_{n,n+m} \\ R_{n+1,n} & [R_{n+1,n+1} - E_0] & \ldots & R_{n+1,n+m} \\ \ldots & & \ldots & \\ R_{n+m-1,n} & R_{n+m-1,n+1} & \ldots & R_{n+m-1,n+m} \\ R_{n+m,n} & R_{n+m,n+1} & \ldots & [R_{n+m,n+m} - E_0] \end{array} \right),$$

which, when factorized,

$$\begin{pmatrix} g_n & a_1 & a_2 & \dots & & a_m \\ a_1^\star & g_{n+1} & b_1 & b_2 & \dots & b_{m-1} \\ \dots & & & & & \dots \\ a_{m-1}^\star & b_{m-2}^\star & \dots & & & z_1 \\ a_m^\star & b_{m-1}^\star & \dots & & & g_{n+m} \end{pmatrix} = \begin{pmatrix} 1 & u_1 & u_2 & \dots & & u_m \\ 0 & 1 & v_1 & v_2 & \dots & v_{m-1} \\ & & \dots & & & \\ 0 & \dots & & & 0 & 1 \end{pmatrix} \times$$

$$\times \begin{pmatrix} f_n^{[n,m]} & 0 & \dots & & 0 \\ 0 & f_{n+1}^{[n,m]} & 0 & \dots & 0 \\ & & \dots & & \\ 0 & & \dots & 0 & f_{n+m}^{[n,m]} \end{pmatrix} \times \begin{pmatrix} 1 & 0 & \dots & & 0 \\ u_1^\star & 1 & 0 & \dots & 0 \\ u_2^\star & v_1^\star & 1 & 0 & \dots \\ & & \dots & & \\ u_m^\star & v_{m-1}^\star & \dots & & 1 \end{pmatrix},$$

define the auxiliary quantities

$$U_n = u_k^{[0,m]}, \quad V_k = u_k^{[1,m]}, \quad W_k = u_k^{[2,m]}, \quad F_k = f_k^{[k,m]}, \quad k = 1,\ 2,\ \dots .$$

Vice versa, reconstruction of the zero-order Hamiltonian may be prescribed by the formula

$$\begin{pmatrix} 1 & U_1 & U_2 & \dots & & U_m & \dots \\ 0 & 1 & V_1 & V_2 & \dots & V_{m-1} & \dots \\ & & \dots & & & & \end{pmatrix} \times$$

$$\times \begin{pmatrix} 0 & 0 & \dots & & & \\ 0 & F_1 & 0 & \dots & & \\ 0 & 0 & F_2 & 0 & \dots & \\ 0 & 0 & 0 & F_2 & 0 & \dots \\ & & & & \dots & \end{pmatrix} \times \begin{pmatrix} 1 & 0 & \dots & & & \\ U_1^\star & 1 & 0 & \dots & & \\ U_2^\star & V_1^\star & 1 & 0 & \dots & \\ U_3^\star & V_2^\star & W_1^\star & & 0 & \dots \\ & \dots & & & & \end{pmatrix}$$

$$= \begin{pmatrix} (H_0)_{00} - E_0 & (H_0)_{01} & (H_0)_{02} & \dots & & \\ (H_0)_{10} & (H_0)_{11} - E_0 & (H_0)_{12} & \dots & & \\ \dots & & & & \dots & \\ (H_0)_{k0} & (H_0)_{k1} & \dots & (H_0)_{kk} - E_0 & (H_0)_{kk+1} & \dots \\ \dots & & & & \dots & \end{pmatrix}$$

which enables us to make the perturbation sufficiently small by the construction itself.

3 QEPT and the zero-order approximants

A common feature of the present perturbative prescriptions lies in the immanent incompleteness of unperturbed spectra. As a technically less trivial example, let us consider a generalized exponential Morse-like oscillator equation [9],

$$[-\frac{d^2}{dr^2}+V(r)]\,\psi(r)=E\,\psi(r),\qquad r\in(-\infty,\infty).$$

$$V(r)=A\,\left(1-\mathrm{e}^{-\mu\,(r-r_\alpha)}\right)^2+B\,\left(1-\mathrm{e}^{-\mu\,(r-r_\beta)}\right)^3+C\,\left(1-\mathrm{e}^{-\mu\,(r-r_\gamma)}\right)^4$$

which, under the Liouvillean [10] change of variables

$$r\to R=R(r)\equiv\exp(-\,r)\in(0,\infty),\qquad \psi(r)\to\phi(R)=R^{1/2}\psi(r)$$

proves equivalent to the more common sextic anharmonic oscillator,

$$[-\frac{d^2}{dR^2}+\frac{L(L+1)}{R^2}+V^{(Liouv.)}(R)]\;\phi(R)=\varepsilon\;\phi(R)$$

$$V^{(Liouv.)}(R)=g_2\,R^2+g_4\,R^4+g_6\,R^6,$$

$$g_2=A\exp(4r_\alpha)+3B\exp(4r_\beta)+6C\exp(4r_\gamma),$$

$$g_4=-\,B\exp(6r_\beta)-4C\exp(6r_\gamma),$$

$$g_6=C\exp(8r_\gamma)>0.$$

$$\varepsilon=2A\exp(2r_\alpha)+3B\exp(2r_\beta)+4C\exp(2r_\gamma)$$

$$L=-\tfrac{1}{2}+\sqrt{A+B+C-E}>-\tfrac{1}{2}.$$

In accord with Singh et al [11], there are no problems with the explicit QE constructions of the separate as well as multiple (i.e., a finite number of) bound states of the latter system. From the purely physical point of view, the Liouvillean transition to the generalized Morse interactions makes the QE zero-order problem less trivial.

3.1 The construction of singlets

In spite of the obvious fact that all the Liouville-equivalent QE wavefunctions preserve the same Singh-like elementary form

$$\phi^{(elementary)}(R)=e^{-G(R)}\;R^{L+1}\sum_{n=0}^{N}h_nR^{2n},\qquad L>-\tfrac{1}{2}.$$

$$G(R) = \frac{1}{4}\lambda R^4 + \frac{1}{2}\xi R^2, \qquad \lambda = \sqrt{C}\exp(4r_\gamma) > 0, \qquad \xi = g_4/(2\lambda)$$

the particular generalized Morse oscillator energies remain only parametrized by a single real angular-momentum-like pramater L,

$$E = A + B + C - Z^2, \qquad Z = L + \tfrac{1}{2} > 0.$$

In a detailed application of the general QE Singh-like prescriptions [12], the insertion of the elementary wavefunction Ansatz converts the differential Schrödinger equation into recurrences,

$$\begin{gathered} B_n h_{n+1} = C_n^{(0)} h_n + C_n^{(1)} h_{n-1}, \\ B_n = 4\,(n+1)\,(n+Z+1), \qquad Z \equiv L + \tfrac{1}{2} \\ C_n^{(0)} = 2\xi(2n+Z+1) - \varepsilon \\ C_n^{(1)} = 2\lambda(2n+Z) - \xi^2 + g_2 \\ n = 0, 1, \ldots . \end{gathered}$$

The termination requirements

$$h_0 \neq 0, \quad h_N \neq 0, \quad h_{N+1} = h_{N+2} = \ldots = 0$$

define then the wavefunctions in the same closed form as in the sextic case above,

$$h_{n+1} = \frac{h_0}{\prod_{k=1}^{n} B_k} \det H^{[n]}, \quad n = 0, 1, \ldots$$

$$H^{[n]} = \begin{pmatrix} C_0^{(0)} & -B_0 & & & \\ C_1^{(1)} & C_1^{(0)} & -B_1 & & \\ 0 & C_2^{(1)} & C_2^{(0)} & -B_2 & \\ & & & \ldots & \\ & \ldots & 0 & C_n^{(1)} & C_n^{(0)} \end{pmatrix}.$$

In accord with Magyari [12], the subsequent insertion of these determinantal formulae in the termination requirements specifies just the QE solvability constraints in an explicit manner.

In our generalized Morse example, the first, scalar Magyari equation reads

$$C_{N+1}^{(1)} \; (= 2\lambda(2N + Z + 2) - \xi^2 + g_2) = 0$$

and remains a simple linear definition, say, of a coupling $g_2 = g_2(z, N)$. We may fix the origin here, $\lambda = 1/2$. The second Magyari condition

$$\det Q^{[N]}(x, y, z) = 0$$

couples then the remaining three free parameters, say, $x\ (= \varepsilon)$, $y\ (= -2\,g_4\ \equiv -\,g_4/\lambda)$ and $z\ (\equiv Z(0)) = (g_4)^2/(2\lambda)^3 - g_2/(2\lambda) - 2$ together.

The multiplet of the QE energies is defined, in contrast to the sextic case, by the explicit formula

$$E = E(z, N) = E_N^{(solvable)} = -4\,N^2 + 4Nz + A + B + C - z^2,$$
$$N = 0, 1, \ldots, N_{max}, \qquad N_{max} = -\,entier[-\,z/2].$$

Thus, the QE energy levels are numbered by the integer N again. Nevertheless, at each particular value of N, we have now a different matrix $Q^{[N]}(x, y, z)$,

$$\left(\begin{array}{ccccc} x + y(Z+1) & 4 + 4Z & & & \\ 2N & x + y(Z+3) & 16 + 8Z & & \\ & \ldots & & & \\ & 4 & x + y(Z + 2N - 1) & 4N^2 + 4NZ & \\ & & 2 & x + y(Z + 2N + 1) & \end{array} \right)$$

and, hence, a different algebraic Magyari equation,

$$\begin{array}{lr} x + y\,(z+1) = 0, & N = 0, \\ x^2 + 2\,y\,z\,x + y^2\,(z^2 - 1) + 8\,(1 - z) = 0, & N = 1, \\ x^3 + y\,x^2\,(3\,z - 3) + x\,[y^2\,(3\,z^2 - 6\,z - 1) + 16\,(5 - 2\,z)] & \\ +y\,(z - 3)\,[y^2\,(z^2 - 1) + 16\,(1 - 2\,z)] = 0, & N = 2, \\ x^4 + y\,x^3\,(4\,z - 8) + x^2\,[y^2\,(6\,z^2 - 24\,z + 14) + 80\,(4 - z)] & \\ +4\,y\,x\,[y^2\,(z^3 - 6\,z^2 + 7\,z + 2) - 8\,(5\,z^2 - 30\,z + 34)] & \\ +(z - 5)\,[y^4\,(z - 3)\,(z^2 - 1) - 16\,y^2\,(5\,z^2 - 15\,z + 4) & \\ +576\,(z - 3)] = 0, & N = 3, \\ \end{array}$$

$$\ldots .$$

Each of these equations possesses exactly $N + 1$ real roots $x = x(y, z)$. Thus, for the clas of potentials in question, the desired zero-order QE bound states always exist and form a two-parameteric family in general.

3.2 The construction of doublets

Let us try to analyse the existence of doublets of the QE states now. Implicitly, they are defined by the pair of coupled Magyari equations

$$\det Q^{[N]}(x,y,z) = 0, \qquad N \geq 0,$$
$$\det Q^{[M]}(x,y,z) = 0, \qquad M > N.$$

From the corresponding menu of the polynomial Magyari equations each of which possesses $N+1$ real roots $x = x(y,z)$ and, simultaneously, $M+1$ real roots $y = y(x,z)$, one has to determine now a mutually selfconsistent doublet solution. Let us proceed constructively.

3.2.1 $N = 0$

Putting $N = 0$ and eliminating $x = -y(z+1)$ in the simplest possible subcase, the answer has an implicit form

$$8\,(1-z) = 0, \qquad M = 1, \qquad z > 2,$$
$$64\,y\,(z-2) = 0, \qquad M = 2, \quad z > 4,$$
$$576\,(3-z)[y^2-(z-5)] = 0, \qquad M = 3, \qquad z > 6,$$
$$2048\,y\,(z-4)\,[3\,y^2-2\,(4\,z-25)] = 0, \qquad M = 4, \qquad z > 8,$$
$$12800\,(5-z)\,[6\,y^4+y^2\,(217-29\,z)+9\,(z-9)\,(z-7)] = 0,$$
$$M = 5, \qquad z > 10,$$
$$221184\,y\,(z-6)\,[5\,y^4+y^2\,(322-37\,z)$$
$$+4\,(8\,z^2-148\,z+675)] = 0, \qquad M = 6, \qquad z > 12,$$
$$\ldots\,.$$

The explicit and closed-form real roots $y = y_m(M,z)$ do exist, indeed,

$$y_1(2,z) = 0,$$
$$y_{1,2}(3,z) = \pm\sqrt{z-5},$$
$$3\,y_{1,2}(4,z) = \pm\sqrt{24\,z-150},$$
$$6\,y_{1,2,3,4}(5,z) = \pm\sqrt{87\,z-651\,\pm 3\,\sqrt{625\,z^2-9130\,z+33481}},$$
$$y_{1,2,3,4}(6,z) = \pm\sqrt{\pm\sqrt{729\,z^2-11988\,z+49684}+37\,z-322}/\sqrt{10},$$
$$\ldots\,.$$

The finite normalization of the wavefunctions must be guaranteed of course. Its explicit condition has a simple form $z > c(M)$ with $c(3) = 5,\ c(4) = 6.25,\ c(5) = 9,\ c(6) = 10.34, \ldots$.

3.2.2 $N = 1$

Putting $N = 1$ and eliminating $\varepsilon(1) = x_{1,2} = -y\,z \pm \sqrt{y^2 + 8\,z - 8}$ with plus or minus sign (i.e., with the two alternative V's), we get

$$8\,(3 - z)\,(3\,\sqrt{y^2 + 8\,z - 8} - y) = 0, \qquad M = 2,$$
$$128\,(z - 4)\,(2\,y\,\sqrt{y^2 + 8\,z - 8} - y^2 - 12) = 0, \qquad M = 3,$$
$$\ldots.$$

$M = 2$ is unacceptable but one may use

$$\sqrt{3}\,y_{1,2}(3, z) = \pm 2\,\sqrt{2\,\sqrt{4\,z^2 - 14\,z + 19} - 4\,z + 7} = \pm 2\,\sqrt{2\,\sqrt{4\,a^2 + 34\,a + 79} - 4\,a - 17}, \quad a = z - 6 > 0$$

etc.

3.3 The construction of triplets

A much longer story of triplets starts with the Magyari set

$$\det Q^{[N]}(x, y, z) = 0, \qquad N \geq 0,$$
$$\det Q^{[M]}(x, y, z) = 0, \qquad M > N,$$
$$\det Q^{[K]}(x, y, z) = 0, \qquad K > M$$

which provides no solutions at $(N, M, K) = (0, 1, K)$ and $(N, M, K) = (1, 2, K)$ and, perhaps (= conjecture) no solutions at any $(N, M, K) = (N, N{+}1, K)$. Also the explicit solutions with $z_i(N, M, K) = z_i(0, 2, 2\,k + 1) = 2k + 2i + 1$ remain unnormalizable. At the same time, with $(N, M, K) = (0, 2, 2\,t)$, $t = 2, 3, \ldots$ and $x = y = 0$, one gets the normalizable

$$q_k\ (\equiv h_{2k}) = (-1)^k 4^{-k} \left(\begin{array}{c} t \\ k \end{array} \right) \frac{\Gamma(1 + Z/2)}{\Gamma(k + 1 + Z/2)} q_0, \quad k = 0, 1, \ldots, t$$

which, as a test, reproduce just the Laguerre polynomials pertaining to the well known ordinary Morse case when properly re-scaled $(\mu \to \mu/2)$.

3.3.1 (0,3,K)

The first two nontrivial and compact triplet-state results read

$$z^{\{1\}} = z_3(0,3,7) = \left(126 + \sqrt{7417}\right)/11 \approx 19.28381877\,(\,>14).$$

$$z^{\{2\}} = z_3(0,3,8) = \left(563\ + 18\,\sqrt{3039}\right)/91 \approx 17.09107983\ (\,>16)$$

and result from quadratic equations. Similarly, further equations read

$$\begin{array}{c} 81\,z^2 - 1694\,z + 5325 = 0, \qquad\qquad K = 9, \\ 374\,z^3 - 20441\,z^2 + 229555\,z + 245840 = 0, \quad K = 10, \\ 1729\,z^4 - 55730\,z^3 - 470604\,z^2 + 20580178\,z - 71914325 = 0, \ K = 11, \\ \ldots . \end{array}$$

Some of them remain solvable nonnumerically and we get

$$\begin{array}{c} z^{\{3\}} = z_4(0,3,10) = \sqrt{160273771}\,\sin[(\theta+\pi)/3]/561 + 20441/1122 \\ \approx 38.0864 > 20, \qquad \mathrm{tg}\theta = 8\,\frac{\sqrt{7262798546165781426915750994030443473 55}}{2427112467348350355585} \end{array}$$

$$z^{\{4\}} = z_5(0,3,11) = \ldots \approx 29.319614 > 22$$

in the first few simplest cases.

3.3.2 (0,4,K)

Mutatis mutandis, one has equations

$$\begin{array}{c} 33649\,z^3 - 1432887\,z^2 + 8607483\,z - 7813045 = 0, \qquad K = 9, \\ 16016\,z^3 - 399768\,z^2 - 1531515\,z + 14532070 = 0, \qquad K = 10, \\ 96135\,z^4 + 45620\,z^3 - 87827630\,z^2 + 696931300\,z - 1172415169 = 0, \\ K = 11, \end{array}$$

and physically acceptable pair of roots

$$\begin{array}{c} z^{\{5\}} = z_2(0,4,9) \\ = 8\,\sqrt{8224066447}\,\sin[(\phi+\pi)/3]/33649 + 477629/33649 \approx 35.57662, \\ \mathrm{tg}\phi = 5\,\frac{\sqrt{164316910812365200329561448669905503500300085 02}}{26064026995931543206 3878}. \end{array}$$

$$
\begin{gathered}
z^{\{6\}} = z_4(0,4,10) \\
= \sqrt{1620838101}\,\sin[(\psi+\pi)/3]/2002 + 16657/2002 \approx 27.24785, \\
\mathrm{tg}\psi = 26217\,\frac{\sqrt{17447214788182041411791148628800 68355}}{5815454354330557 1091730}. \\
z^{\{7\}} = z_5(0,4,11) = \ldots \approx 25.153897 > 22. \ldots .
\end{gathered}
$$

3.3.3 (1,3,K)

From

$$
\begin{gathered}
7075\,z^3 - 15\,z^2\,(80\,v + 5739) + 15\,z\,(796\,v + 19651) \\
-24780\,v - 271915 = 0, \quad K = 4, \\
\ldots
\end{gathered}
$$

one eliminates $v = \sqrt{64(4z^2 - 14z + 19)/9}$ and gets

$$
\begin{gathered}
13475\,z^4 - 196000\,z^3 + 725114\,z^2 - 659040\,z - 272925 = 0, \quad K = 4, \\
560\,z^3 - 8351\,z^2 + 33096\,z - 33260 = 0, \qquad K = 5, \\
\ldots,
\end{gathered}
$$

with our final non-numerical parameter

$$
z^{\{8\}} = z_2(1,3,4) = \ldots \approx 9.3974688 > 8.
$$

and its many further numerical descendants,

$$
z^{\{9\}} = z_2(1,3,6) = \ldots \approx 15.07882 > 12
$$

(an eight-degree polynomial needed) etc.

4 Summary

We reviewed our two recent perturbative constructions of bound states. With a methodical guide and inspiration being found in the simple Schrödinger - Gallas example, we succeeded in a reduction of all the necessary mathematics to the solution of the nonlinear (coupled, polynomial) algebraic equations. With a thorough assistance of the computerized symbolic manipulations, our first (Rayleigh-Schrödinger-inspired) QEPT prescription and its FPPT modification have been applied to the standard double-well Gallas oscillator.

A rich structure of the QE solutions has also been discovered for the more realistic generalized Morse exponential oscillators. The existence as well as an utterly unexpected simplicity of the corresponding QE singlets, doublets and triplets of non-numerical (!) solutions seems to be a highly challenging open question now. The traditional simple-minded Lie-algebraic explanation of their existence, structure and normalizability seems to fail.

In the latter, technically less trivial example, the FPPT construction was not finished yet. Nevertheless, as long as the applicability of a fixed-point perturbation theory seems to reach far beyond the scope of the related higher-order QEPT constructions (which, traditionally, would employ just the analytic and/or generalized infinite continued fractions as an auxiliary semi-numerical tool), we have also reviewed here the general FPPT scheme of construction of the (asympotic) "best effective Hamiltonians", without any infinite continued fraction expansions. The basic idea of their present FPPT replacement by some finite-expansion quantities seems promising – its "small" perturbation parameter is artificial and coincides, roughly speaking, with the inverse model-space dimension.

References

[1] M. Znojil, J. Phys. A: Math. Gen. 27 (1994) 4945-68 and 7491-501

[2] M. Znojil, Cz. J. Phys. 37 (1987) 1072-8 and 44 (1994) 545-56

[3] M. Tater, J. Phys. A 20 (1987) 2483

[4] F. M. Fernández and R. Guardiola, J. Phys. A 26 (1993) 7169

[5] R. Bishop, Theor. Chim. Acta 80 (1991) 95

[6] J. Gallas, J. Phys. A 21 (1988) 3393, with further references

[7] R. Whitehead et al, J. Phys. A 15 (1982) 1217

[8] M. Znojil, in "Selected topics in QFT", ed. by J. Niederle and J. Fischer (World Scientific, London, 1990), pp. 376 - 380

[9] J. Čížek et al, J. Chem. Phys. 99 (1993) 7331

[10] J. Liouville, J. Math. Pures Appl. 1 (1837) 16

[11] V. Singh et al, Phys. Rev. D 18 (1978) 1901

[12] E. Magyari, Phys. Lett. A 81 (1981) 116; M. Znojil and P. G. L. Leach, J. Math. Phys. 33 (1992) 2785-94