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Circular vectors and toroidal matrices

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Abstract

Arrays of numbers may be written not only on a line (= "a vector") or in the plain (= "a matrix") but also on a circle (= "a circular vector"), on a torus (= "a toroidal matrix") etc. In the latter cases, the imanent index-rotation ambiguity converts the standard "scalar" product into a binary operation with several interesting properties.

1 Motivation

In the applied quantum physics[1], wavefunctions $\varphi_i, \quad i = \ldots, -1, 0, 1, \ldots$ often appear restricted by the periodic boundary conditions

$$\varphi_i = \varphi_j, \qquad i \equiv j (\mathrm{mod} N). \tag{1}$$

Similarly, square matrices with a double cyclic (or toroidal) symmetry

$$\Omega_{i,j} = \Omega_{k,l}, \quad i \equiv k (\mathrm{mod} N), \quad j \equiv l (\mathrm{mod} N)$$

may be introduced *per analogiam*.

Cyclic symmetry (1) rarely transcedes the role of a technical trick which simplifies computations. Here, we shall pay attention to its further properties which seem to parallel some geometric aspects of various hypercomplex numbers[2].

[1] M. Znojil, Phys. Rev. B40 (1989) 12468-75

[2] cf. J. Bečvář, Pokr. mat. fyz. astr. 38 (1993) 305-17 or any other recent review of the further related literature)

2 A scalar-product-like binary operation

In practice, there might exist an N-tuple ambiguity of a conversion of the cyclic or circular vectors (1) (let us denote them by a super-circle, $\mathring{\varphi}$) into their row and/or column predecessors, say,

$$(\varphi_{k+1},\ \varphi_{k+2},\ \ldots,\ \varphi_{k+N}).$$

Thus, whenever k may be arbitrary, an overlap (= scalar product) of any pair $\mathring{\varphi}$ and $\mathring{f}$ also remains ambiguous. Thus, the circular symmetry is a source of the whole family of scalar products. We may arrange them in a set

$$t_{j+k} = \sum_{i=1}^{N} f_i\,\varphi_{i+j}, \quad j = 0, 1, \ldots, N-1, \quad k = \text{arbitrary}. \tag{2}$$

or, alternatively,

$$h_{k+j} = \sum_{i=1}^{N} f_{i+k}\,\varphi_i, \quad k = 0, 1, \ldots, N-1, \quad j = \text{arbitrary}.$$

Here, we shall pick up the former case for the sake of definitness, and denote our "product" by a small square, $\mathring{t} = \mathring{f}\Box\mathring{\varphi}$.

Similarly, in the case of toroidal matrices, we may contemplate any one of the following four quasi-scalar products

$$O_{k+p,l+r} = \sum_{i=1}^{N}\sum_{j=1}^{N} Z_{i+k,j+l}\,\Omega_{j,i}, \qquad P_{k+p,l+r} = \sum_{i=1}^{N}\sum_{j=1}^{N} Z_{i+k,j}\,\Omega_{j+l,i}$$

$$Q_{k+p,l+r} = \sum_{i=1}^{N}\sum_{j=1}^{N} Z_{i,j+k}\,\Omega_{j,i+l}, \qquad R_{k+p,l+r} = \sum_{i=1}^{N}\sum_{j=1}^{N} Z_{i,j}\,\Omega_{j+k,i+l} \tag{3}$$

with $k, l = 0, 1, \ldots, N-1$ and arbitrary p and r, etc.

3 Parametrizations

3.1 The simplest example – circular vectors with N=2

At $N = 2$, circular symmetry (1) means that we have to deal with the unordered pairs of numbers, $\mathring{f} = (a,\, b)$. Their parametrization $\mathring{f} = (m\,\mathrm{ch}t,\, m\,\mathrm{sh}t)$

induces a new (bracketed, ordered) denotation $\overset{\circ}{f} \equiv [m,\, t]$ and $\overset{\circ}{\varphi} \equiv [\mu,\, \tau]$ and simplifies their present "multiplication",

$$[m,\, t]\Box[\mu,\, \tau] = [m\mu, t+\tau].$$

This seems inspiring: The set of all the nonzero elements $\overset{\circ}{f} = (a,\, b),\ a \neq \pm b$ (i.e., $m \neq 0$) forms a commutative and associative group with the unit $[1, 0]$ (i.e., $(1,\, 0) \equiv (0,\, 1)$) and with the trivial inversion $\overset{\circ}{f}^{-1} = [1/m,\, -t]$.

In the $a-b$ plane, the $N = 2$ circularity of vectors $\overset{\circ}{f} = (a,\, b)$ introduces a symmetry $a \leftrightarrow b$ which unites hyperbolas $(m\,\mathrm{ch}t,\, m\,\mathrm{sh}t)$ and $(m\,\mathrm{sh}t,\, m\,\mathrm{ch}t)$, $t \in (-\infty, \infty)$. Their simultaneous rotation is mediated by the elements $[1,\, -\tau]$.

3.2 The first nontrivial case with N=3

The cyclically permutable vectors $\overset{\circ}{f} = (a,\, b,\, c)$ restricted by the invertibility condition

$$a^3 + b^3 + c^3 - 3abc \neq 0$$

may be parametrized, say, in accord with the formula

$$\begin{array}{c} 3a = m\exp t + 2m\exp(-\frac{1}{2}t)\cos s \\ 3b = m\exp t + 2m\exp(-\frac{1}{2}t)\cos(s + \frac{2}{3}\pi) \\ 3c = m\exp t + 2m\exp(-\frac{1}{2}t)\cos(s + \frac{4}{3}\pi). \end{array} \tag{4}$$

In the new notation $\overset{\circ}{f} = [m,\, t,\, s]$ the simplicity and transparency of the multiplication law (2),

$$[m,\, t,\, s]\Box[\mu,\, \tau,\, \sigma] = [m\,\mu,\, t+\tau,\, -s+\sigma]$$

reveals its non-commutativity and non-associative character,

$$\begin{array}{c} [M,\, T,\, S]\Box\{[m,\, t,\, s]\Box[\mu,\, \tau,\, \sigma]\} = [M\,m\,\mu,\, T+t+\tau,\, -\Sigma - s + \sigma] \\ \{[M,\, T,\, S]\Box[m,\, t,\, s]\}\Box[\mu,\, \tau,\, \sigma] = [M\,m\,\mu,\, T+t+\tau,\, +\Sigma - s + \sigma]. \end{array}$$

The left unit is $\overset{\circ}{u} = [1, 0, 0]$ but there exists no right unit. Thus, even though the inverse elements exist whenever $m \neq 0$, our multiplication only forms a groupoid.

In a way parallelling the $N = 2$ case where zeros form the pair of lines $a = \pm b$, the $N = 3$ zeros lie not only in the analogous plane $a+b+c = 0$ but also on the line $a = b = c$. With these zeros removed, the remaning three-dimensional space may move under the left or right action of its elements.

Due to the absence of right units, a preservation of any element by the right action is not possible. In this sense, the left-unit elements [1, 0, 0] may only be re-interpreted as certain involutions or "square-rooted" right units.

The $N = 3$ analogues of the $N = 2$ hyperbolas are surfaces defined by their elliptic intersections with certain planes. Indeed, we have, identically,

$$a^3 + b^3 + c^3 - 3abc \equiv \tfrac{1}{2}(a + b + c)\{(a - b)^2 + (a - c)^2 + (b - c)^2\}$$

where, by our above definition (4), we have $a + b + c = \exp t$ etc.

3.3 Circular vectors with N=4

As long as the $N = 4$ "zeros" $m = 0$ reflect just a disappearance of the determinant

$$\det \begin{pmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{pmatrix} = (a + b + c + d)(a - b + c - d)\{(a - c)^2 + (b - d)^2\}$$

we may parametrize the whole four-dimensional space and its motion via hyper-planes

$$a + b + c + d = \exp(t + s), \quad a - b + c - d = \exp(t - s)$$

and hyper-ellipsoids

$$\{(a - c)^2 + (b - d)^2\} = \exp(-2t).$$

Thus, with $a - c = \cos r$, $\mathring{f} = [m, t, s, r]$ and multiplication rule

$$[m,\, t,\, s,\, r]\Box[\mu,\, \tau,\, \sigma,\, \rho] = [m\,\mu,\, t + \tau,\, s + \sigma,\, -r + \rho]$$

the motion of the space splits in the multiplicative and rotational parts again.

3.4 Playing games with the higher N's

An extension of all the above constructions to $N > 4$ is more difficult: At all the N's we have

$$\det \begin{pmatrix} a & b & c & \ldots & y & z \\ z & a & b & \ldots & x & y \\ & & & \ldots & & \\ b & c & d & \ldots & z & a \end{pmatrix} = (a+b+c+\ldots+z)\, D(a,b,\ldots,z)$$

but $D(\ldots)$ does not always decompose easily. Even in a maximally reduced case with zero parameters $c = d = \ldots = 0$, it is rather difficult to get the factorization

$$D(a,b,0,0,0) = a^4 - a^3 b + a^2 b^2 - a\,b^3 + b^4$$

$$= \frac{5}{16}\left[(a-b)^2 + \left(1 - \frac{2}{\sqrt{5}}\right)(a+b)^2\right]\left[(a-b)^2 + \left(1 + \frac{2}{\sqrt{5}}\right)(a+b)^2\right].$$

Similar difficulties emerge also at $N = 7$ etc.

The question of feasibility of the underlying algebraic manipulations is more challenging at the even N's. Thus, we get

$$D(a,b,c,d,e,f) = (a-b+c-d+e-f) \times E_+(a,b,c,d,e,f) E_-(a,b,c,d,e,f)$$

with

$$2\,E_\pm(a,b,c,d,e,f) = \pm[(a-b)^2+(b-c)^2+(c-d)^2+(d-e)^2+(e-f)^2+(f-a)^2]$$

$$+[(a-c)^2+(b-d)^2+(c-e)^2+(d-f)^2+(e-g)^2+(f-b)^2] \mp 2\,[(a-d)^2+(b-e)^2+(c-f)^2]$$

at $N = 6$, and

$$D(a,b,\ldots,h) = (a-b+c-d+e-f+g-h) \times E(a,b,c,\ldots,h) F(a,b,c,\ldots,h)$$

with

$$E(a,b,c,\ldots,h) = -[(a-e)^2 + (b-f)^2 + (c-g)^2 + (d-h)^2]$$

$$+[(a-c)^2+(b-d)^2+(c-e)^2+(d-f)^2+(e-g)^2+(f-h)^2+(g-a)^2+(h-b)^2]$$

and

$$F(a,b,c,\ldots,h) = (\alpha^2 - \gamma^2 + 2\beta\delta)^2 + (\beta^2 - \delta^2 - 2\alpha\gamma)^2$$

with $\alpha = a - e,\ \beta = b - f,\ \gamma = c - g$ and $\delta = d - h$ at $N = 8$.

3.5 Toroidal matrices at $N=2$

Let us pick up, say, the Q-type products (3) of the doubly cyclic matrices, and re-write their $N=2$ realization

$$\left(\begin{array}{cc} A & B \\ C & D \end{array}\right) = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \Box \left(\begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array}\right)$$

in the standard linear-algebraic form

$$\left(\begin{array}{c} A \\ B \\ C \\ D \end{array}\right) = \left(\begin{array}{cccc} a & c & b & d \\ c & a & d & b \\ b & d & a & c \\ d & b & c & a \end{array}\right) \times \left(\begin{array}{c} \alpha \\ \beta \\ \gamma \\ \delta \end{array}\right).$$

Now, in a close parallel to the case of vectors at $N=4$, we may put $a+b+c+d=\exp(t+r)$, $a-b+c-d=\exp(t-r)$ and $a+b-c-d=\exp(-t+s)$, $a-b-c+d=\exp(-t-s)$. In the new parametrization and notation,

$$\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) = [[m,\, t,\, r,\, s]]$$

the entirely straightforward manipulations confirm that

$$[[m,t,r,s]]\Box[[\mu,\tau,\rho,\sigma]] = [[m\mu, \tau+\tfrac{1}{2}r+\tfrac{1}{2}s, t+\rho+\tfrac{1}{2}r-\tfrac{1}{2}s, t+\sigma-\tfrac{1}{2}r+\tfrac{1}{2}s]] \quad (5)$$

We may conclude that the non-existence of the right unit survives the transition to matrices.

In the standard linear algebra language, the "angular" part of the product (5) reads

$$\left(\begin{array}{c} T \\ R \\ \Sigma \end{array}\right) = U \left(\begin{array}{c} t \\ r \\ s \end{array}\right) + \left(\begin{array}{c} \tau \\ \rho \\ \sigma \end{array}\right), \qquad U^2 = I, \qquad U = \left(\begin{array}{ccc} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & \frac{1}{2} & -\frac{1}{2} \\ 1 & -\frac{1}{2} & \frac{1}{2} \end{array}\right).$$

In contrast to the similar four-dimensional vectorial case, the new "imaginary-unit-like" square root of the identity U becomes non-diagonal.