Strong-coupling expansions for the $\mathcal{PT}$-symmetric oscillators $V(x) = a\,(ix) + b\,(ix)^2 + c\,(ix)^3$

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Abstract

We study the traditional problem of convergence of perturbation expansions when the hermiticity of the Hamiltonian is relaxed to a weaker $\mathcal{PT}$ symmetry. An elementary and quite exceptional cubic anharmonic oscillator is chosen as an illustrative example of such models. We describe its perturbative features paying particular attention to the strong-coupling regime. Efficient numerical perturbation theory proves suitable for such a purpose.

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1 Introduction and summary

The analysis of convergence of perturbation expansions is one of the most exciting mathematical problems in quantum physics. Its relevance ranges from the phenomenology of bound states in atomic and molecular physics up to abstract methodical considerations which concern quantum fields [1]. Often, the convergence of perturbation series is studied by means of an elementary anharmonic oscillator model in one dimension,

$$V(x) = B\,x^2 + C\,x^3 + D\,x^4, \qquad x \in (-\infty, \infty), \tag{1}$$

because of its apparent simplicity. The application of textbook Rayleigh-Schrödinger (RS) perturbation theory is remarkably simple in the so called weak coupling regime dominated by harmonic oscillations when $B \gg |C|$ and $B \gg |D|$.

In more sophisticated approaches, based on an extensive numerical re-processing of the RS series, one considers the strong-coupling regime in which the quartic term is not only important but also dominating [2, 3]. This is of particular importance in field theory where perturbation expansions are one of the few available constructive means. The strong-coupling approach may also be of value in a quantitative description of seemingly non-perturbative quantum mechanical systems of physical interest [4].

Allowing complex values of the couplings of model (1) further enriches its physical meaning and scope [5]. It opens new phenomenological possibilities. Energies may acquire purely imaginary components, bound states may "dissolve" into unstable resonances, etc. Various mathematical techniques have to be combined in order to treat these extended models with non-hermitean Hamiltonians – cf. ref. [6] for a review of some relevant literature. In particular, Bender and Boettcher [7] recently discovered, among all the complex quartic polynomial potentials (1), a large two-parametric family of non-hermitean and $\mathcal{PT}$-symmetric forces which are partially, *quasi-exactly* [8] solvable.

The parity-violating Bender and Boettcher example is already slightly counterintuitive and exceptional. Strictly speaking, a rigorous treatment of the similar quartic potentials and, in general, of all their higher-power $\mathcal{PT}-$symmetric perturbations $(ix)^N$ with $N \geq 4$ already requires a nontrivial deformation of the integration path into the complex plane [9]. A slight formal simplification is achieved when the dominant quartic anharmonicity in eq. (1) vanishes [10]. In this setting, trying to bridge the gap between the hermitean and non-hermitean Hamiltonians in what follows, we will analyze just the non-hermitean and $\mathcal{PT}$-symmetric shifted version of eq. (1) with $D = 0$,

$$V(x) = a\,(ix) + b\,(ix)^2 + c\,(ix)^3, \qquad x \in (-\infty, \infty). \tag{2}$$

Historically, the empirical and numerically oriented studies of interactions (2) with $a = 0$ go back to Daniel Bessis who conjectured, several years ago, that the purely imaginary cubic coupling keeps the "resonant" anharmonic energies discrete and, what is even more surprising, *purely real* [11]. This property makes these models immediately eligible for perturbative description.

A new, additional motive for perturbative analysis of non-hermitean oscillators of Bessis type lies in unexpected difficulties related to their weak coupling perturbative interpretation. Indeed, one cannot allow the cubic *and* quadratic coupling vanish simultaneously since the spectrum of linear $V_0(x) = ix$ is null [12]. In this sense, besides a far reaching analogy between eqs. (1) and (2) (which was the original inspiration of our present paper) there also exist certain differences.

Several particular cases of potential (2) with real couplings are discussed here. In section 2 we recall the ideas of numerical perturbation theory [13, 14] and show that its application to the class of non-hermitean examples (2) is straightforward. In section 3, we add more observations regarding perturbative expansions for our $\mathcal{PT}$ symmetric Hamiltonians, mention their possible methodical connection to field theory etc.

We may summarize that even our utterly schematic examples confirm that many $\mathcal{PT}$-symmetric non-hermitean oscillators similar to eq. (2) may be understood and described in a way which strictly parallels the existing extensive studies of the ordinary anharmonic oscillator (1). The continuation of real Hamiltonians to complex plane with preservation of their $\mathcal{PT}$ symmetry opens a new and almost unexplored field of mathematical analysis of the Schrödinger equation.

We have resorted to the simple numerical algorithm to compute perturbation series. Such a numerical form of perturbation theory proved convenient as a stable source of expansions that are suitable for all values of the coupling constant. In their strong coupling limit as well as renormalized form they were shown to offer a reliable computational tool. A possibility of using a virtually arbitrary zero-order system is well matched by the "perturbation friendly" real and discrete character of spectra of all the $\mathcal{PT}-$symmetric models of the Bessis type. Their eigenfunctions are complex valued, with the only requirement of giving square integrable wavefunctions of course. One may appreciate the role of the $\mathcal{PT}$ symmetry of Hamiltonians which do not only provide *real* eigenvalues but also their perturbation form with *real* coefficients.

2 Perturbation series by numerical techniques

2.1 Formalism

Traditional textbooks on quantum mechanics pay a thorough attention to the construction of perturbative expansions based on the expansion in a complete set of eigenvectors of the unperturbed Hamiltonian H_0. For this reason, the perturbation series

$$E = E_0 + \lambda\, E_1 + \lambda^2\, E_2 + \dots\ . \tag{3}$$

for the eigenvalues of Hamiltonians $H(\lambda) = H_0 + \lambda W$ are commonly restricted to a small vicinity of a few available exactly solvable models.

In contrast, a purely numerical approach is both straightforward and more widely applicable to the determination of perturbation expansions like (3). More precisely, a simple and efficient procedure for obtaining RS perturbation expansions for the solutions of the Schrödinger equation with Hamiltonians

$$H = -\frac{d^2}{dx^2} + V(x) + \lambda W(x) \tag{4}$$

consists in enclosing the system in a box $[X_{\min}, X_{\max}]$ with impenetrable walls at the end points [14]. Then one discretizes $V_n = V(X_{\min} + nh)$ (plus a similar formula for W_n) and replaces the second derivative operator by the centered second differences

$$d^2/dx^2 \approx \delta^2/h^2 \ ,$$

h being the distance between adjacent points in a uniform mesh. The problem reduces to the diagonalization of a *symmetric tridiagonal* matrix, with diagonal elements given by $V_n + \lambda W_n + 2h^{-2}$ and upper and lower diagonals with constant matrix elements $-1/h^2$.

In the case of complex potentials, the only difference is that both V_n and W_n may be complex quantities. From now on, capital and lower case boldface letters denote square matrices and column vectors, respectively.

As indicated above, we solve the matrix eigenvalue problem

$$[\boldsymbol{H}_0 + \lambda \boldsymbol{H}_I]\boldsymbol{v} = E\boldsymbol{v}$$

where the Hamiltonian matrix $\boldsymbol{H}_0 + \lambda \boldsymbol{H}_I$ is symmetric, with

$$\boldsymbol{H}_0 = \begin{bmatrix} \frac{2}{h^2} + V_0 & -\frac{1}{h^2} & 0 & 0 & \cdots \\ -\frac{1}{h^2} & \frac{2}{h^2} + V_1 & -\frac{1}{h^2} & 0 & \cdots \\ 0 & -\frac{1}{h^2} & \frac{2}{h^2} + V_2 & -\frac{1}{h^2} & \\ \vdots & & \ddots & \ddots & \ddots \end{bmatrix}$$

and

$$\boldsymbol{H}_I = \begin{bmatrix} W_0 & 0 & \cdots \\ 0 & W_1 & \\ \vdots & & \ddots \end{bmatrix} .$$

When applying perturbation theory one first solves the eigenvalue equation for the unperturbed problem

$$\boldsymbol{H}_0\boldsymbol{v}_0 = E_0\boldsymbol{v}_0 \tag{5}$$

and then a hierarchy of inhomogeneous equations for the perturbation corrections of higher order,

$$\boldsymbol{H}_0\boldsymbol{v}_k + \boldsymbol{H}_I\boldsymbol{v}_{k-1} = E_0\boldsymbol{v}_k + E_k\boldsymbol{v}_0 + \sum_{p=1}^{k-1} E_p\boldsymbol{v}_{p-k} \ . \tag{6}$$

Notice that $\boldsymbol{v}_0$ is both a left and right eigenvector because of the symmetry of $\boldsymbol{H}_0$. We find it convenient to use the scalar product $[\boldsymbol{v}^T\boldsymbol{w}] \equiv \sum v_i w_i$, although it does not produce a real norm, and resort to the *intermediate normalization* condition

$$[{\boldsymbol{v}_0}^T\boldsymbol{v}_k] \equiv \sum_i v_{0i}v_{ki} = \delta_{k0}$$

that leads to simpler expressions for the energy coefficients. The intermediate normalization condition is based on the observation that if $\boldsymbol{v}_k$ is a solution of the perturbation equation, then $\boldsymbol{v}_k + \alpha \boldsymbol{v}_0$ is also a solution for arbitrary values of α. The computational scheme is straightforward:

1. Solve the eigenvalue equation (5)

2. Project (6) onto ${\boldsymbol{v}_0}^T$ and obtain the k–th perturbative correction to the energy

$$E_k = [{\boldsymbol{v}_0}^T \boldsymbol{H}_I \boldsymbol{v}_{k-1}]$$

3. Once E_k is known, solve the inhomogeneous equation for $\boldsymbol{v}_k$

4. Orthogonalize $\boldsymbol{v}_k$ with respect to $\boldsymbol{v}_0$,

$$\boldsymbol{v}_k \leftarrow \boldsymbol{v}_k - [{\boldsymbol{v}_0}^T \boldsymbol{v}_k] \boldsymbol{v}_0 .$$

2.2 Application to the two $\mathcal{PT}$ symmetric models

Table 1 shows some perturbation corrections to the energies of the $\mathcal{PT}$ symmetric Hamiltonians

$$H^{(1)} = p^2 + ix^3 + i\lambda x \tag{7}$$

and

$$H^{(2)} = p^2 + ix^3 + \lambda x^2 \ . \tag{8}$$

In both cases the coefficients $E_n^{(1,2)}$ of the powers of λ are *real*, as expected from the $\mathcal{PT}$ symmetry. We used complex arithmetics for the numerical calculation observing that the imaginary parts vanished within the rounding errors. The code was successfully tested with the exactly solvable model

$$H = p^2 + x^2 + i\,\lambda x \tag{9}$$

obtaining the known expansion

$$E(\lambda) = 1 + \lambda^2/4$$

that one easily derives by means of the coordinate translation $x \to x + i\lambda/2$ in the Hamiltonian $H = p^2 + x^2$. The agreement between the numerical and exact result suggests that the method is sufficiently accurate and stable.

The two perturbative expansions of Table 1 are connected because the Hamiltonians (7) and (8) are related by a complex translation of the coordinate. The substitution $x \to x + i\lambda/3$ in eq. (8) results in a Hamiltonian like eq. (7):

$$H \to p^2 + ix^3 + \left(\frac{i\lambda^2}{3}\right) x - \frac{2}{27}\lambda^3 .$$

Consequently, the energy coefficients $E_n^{(1)}$ and $E_n^{(2)}$ of Table 1 satisfy

$$E_{2n}^{(2)} = \frac{1}{3^n} E_n^{(1)} , \qquad E_{2n+1}^{(2)} = -\frac{2}{27}\delta_{n1} \tag{10}$$

indicating that the expansion $E_n^{(2)}$ has only even coefficients except for $E_3^{(2)} = -2/27$. Our numerical coefficients shown in Table 1 obey the exact relations (10) with an accuracy close to 1 part in 10^{10}. This test is an additional confirmation of the stability and accuracy of our numerical method.

3 The overall features of convergence

From the magnitude of the energy coefficients in the tests of the preceding section we may estimate the radii of convergence to be about 1.7 and 3 for the Hamiltonians (7) and (8), respectively. It is sometimes possible to extend the perturbative prediciton beyond these limits.

3.1 Renormalized perturbation expansions

One of the most common techniques of improvement of convergence of a perturbation series is its renormalization. It consists in a nonlinear mapping of the original perturbation parameter (λ in equations above) onto a more convenient one. There are many equivalent mappings (see e.g., Ref. [3]). Here we consider one which has lately received great attention [15, 16] in connection to strong–coupling expansions. It changes the perturbation parameter $\lambda \in [0, \infty]$ into $\kappa \in [0, 1]$ according to

$$\lambda^{5/4} = \frac{\omega(1-\kappa)^{5/4}}{\kappa} \tag{11}$$

where the exponent 5/4 is appropriate for the Hamiltonian in eq. (8), and ω is a free parameter. Application of the scale transformation $x \to x\sqrt{1-\kappa}$ to the Hamiltonian (8) leads to the unitarily equivalent Hamiltonian

$$H^{(r)} = \sqrt{\frac{\lambda}{1-\kappa}} \left[p^2 + (1-\kappa)\left(x^2 - \frac{ix^3}{\omega} \right) + \frac{ix^3}{\omega} \right], \tag{12}$$

where $(1-\kappa)$ is the new expansion parameter.

The algorithm used to calculate the original perturbation expansion also applies to the renormalized series given by the Hamiltonian (12). The resulting expansion coefficients are shown in Table 2. After searching for an optimal value of ω we chose $\omega = 1$ in all our calculations.

Although the coefficients of the renormalized series decrease with the order more slowly than the coefficients of the standard expansion, the former series exhibits better convergence properties because the new perturbation parameter $(1-\kappa)$, is limited to the interval $[0, 1]$. In fact, the relative difference between the renormalized series for the ground–state energy (using the coefficients in Table 2) and the eigenvalue obtained by direct numerical integration of the Schrödinger equation is always smaller than 10^{-5}, even for huge values of λ, as shown in Table 3. Clearly, renormalization is successful in this case.

We may conclude that the application of a numerical version of perturbation theory to non-hermitean Hamiltonians is straightforward as long as their spectrum is discrete. The demonstrated feasibility of a reliable *quantitative* perturbative description of our unusual model (2) with $\mathcal{PT}-$symmetric forces is encouraging. In the language of perturbation theory the phenomenon of existence of real energies becomes more transparent.

3.2 A schematic interpretation of convergence

The renormalized coefficients in Table 2 exhibit a surprising oscillatory behavior with an amplitude that decreases slowly with the perturbation order. We find this resummation-friendly behavior remarkable as it does not occur in similar expansions for real Hamiltonians [16]. Let us now show that there exists an interesting connection of this feature of the one-dimensional quantum mechanics not only with their possible generalizations in field theory [1, 6] but also, in an opposite direction, with their certain schematic zero-dimensional "predecessors".

In a similar way mixing the quantum mechanical and field theoretical perspective people usually analyze the parity-preserving models (1) with $V(\varphi) = \varphi^2/2 + g\,\varphi^4$. Our example (2) is less common, even though it is closely related to effective interactions of physical interest [10]. Still, a parallel seems to exist between the $\mathcal{PT}-$ symmetry on one hand and the hermitean complex conjugation of the underlying Schrödinger equation on the other hand. Such a conjecture is clearly supported by our numerical results, especially when we keep in mind the reinterpretation of quantum mechanics as a model of field theory in a particular *one-dimensional* space-time [1].

Let us now contemplate field theory with Lagrangians $V(\varphi) = \frac{\omega^2}{2}\varphi^2 + \mu\,\varphi^3$ in a *zero-dimensional* space-time. After such a drastic reduction of the phase space the current path integrals of field theory degenerate to ordinary integrals. It is extremely rewarding to recall their perturbation expansions and analyze the similarities in the convergence pattern between the zero and one dimensions.

In the common quartic anharmonic oscillator case one may rely upon the extensive literature. The quantity of interest is the ordinary integral

$$Z(g) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-(\varphi^2/2 + g\varphi^4)} \, d\varphi \, . \tag{13}$$

In our highly schematic zero-dimensional space-time this Feynman-Kac formula is easily given a perturbative form. The resulting weak-coupling perturbation series

$$Z(g) = \sum_{k=0}^{\infty} \frac{(-4g)^k}{\sqrt{\pi}} \frac{\Gamma(2k+1/2)}{k!} \tag{14}$$

is strongly divergent. This exemplifies several mathematical features of interest. Besides the overall parallels with the more-dimensional cases, one may also apply the current Borel resumation technique more immediately. As long as integral (13) may be expressed in a closed form proportional to the modified Bessel function $K_{1/4}(x)$ (cf. formula 3.323.3 in ref. [17]) we may better understand all details. In particular, we may re-construct our quantity $Z(g)$ from its divergent perturbative representation (14) and verify that the reconstruction is correct.

Without a hermiticity of the interaction we may contemplate a replacement of the quartic Lagrangian by our present models. Among all the eligible purely imaginary and spatially antisymmetric candidates $i\,\sum_{j=0}^{J} w_j\,\varphi^{2j+1}$ we shall pick up just the two-term forces with $J=1$ and with the real couplings $w_0 = g > 0$ and $w_1 = f > 0$. They seem instructive enough. In a way paralleling the previous hermitean case, the $\mathcal{PT}-$symmetric alternative to eq. (13), i.e., quantity

$$Z(g,f) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\,(g\,\varphi+f\varphi^3)}\;d\varphi, \tag{15}$$

is again easily represented by the usual perturbation series. In the weak coupling limit $f \to 0$ one encounters the first surprise. Taking the available closed formula Nr. 3.713.2. from ref. [17] one gets

$$Z(g,f) = \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-f)^k}{g^{3k+1}}\frac{\Gamma(3k+1)}{k!}\cos[(k+1/2)\pi]. \tag{16}$$

This series becomes identically equal to zero. One recognizes a clear zero-dimensional parallel to the nullity of spectrum in one dimension.

In an opposite extreme, at very small values of the coupling g the situation is entirely different. The new perturbation series generated in a small vicinity of the Bessis-like massless cubic force forms another counterpart of eq. (14),

$$Z(g,f) = \frac{1}{3\pi}\sum_{k=0}^{\infty}\frac{(-g)^k}{f^{(k+1)/3}}\frac{\Gamma[(k+1)/3]}{k!}\cos[(k/3-1/6)\pi]. \tag{17}$$

With a weakly coupled linear term, it is well convergent. Its detailed form of coefficients mimics the sign-changing pattern of Table 2. Finally, the fact of its convergence itself also parallels the same property of the strong-coupling solutions of the more standard and parity-preserving quartic anharmonic models.

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Table 1: Energy coefficients for the ground states of the Hamiltonians (7) and (8).

	$p^2 + ix^3 + i\lambda x$		$p^2 + ix^3 + \lambda x^2$	
n	$E_n^{(1)}$		$E_n^{(2)}$	
0	1.156267071982		1.156267071982	
1	0.590072533078		0	
2	0.119413858091		0.196690844356	
3	-1.142128489951	$\times 10^{-2}$	-7.407407406938	$\times 10^{-2}$
4	2.197147795152	$\times 10^{-3}$	1.326820645262	$\times 10^{-2}$
5	-5.299287434734	$\times 10^{-4}$	0	
6	1.432651458960	$\times 10^{-4}$	-4.230105523351	$\times 10^{-4}$
7	-4.148903908710	$\times 10^{-5}$	0	
8	1.257939817338	$\times 10^{-5}$	2.712528133232	$\times 10^{-5}$
9	-3.941134497194	$\times 10^{-6}$	0	
10	1.265478160326	$\times 10^{-6}$	-2.180776735057	$\times 10^{-6}$
11	-4.141789581668	$\times 10^{-7}$	0	
12	1.376462051952	$\times 10^{-7}$	1.965228322941	$\times 10^{-7}$
13	-4.632126852199	$\times 10^{-8}$	0	
14	1.575200873072	$\times 10^{-8}$	-1.897075421079	$\times 10^{-8}$
15	-5.404290239370	$\times 10^{-9}$	0	
16	1.868293677263	$\times 10^{-9}$	1.917298897845	$\times 10^{-9}$
17	-6.501605918143	$\times 10^{-10}$	0	
18	2.275683198987	$\times 10^{-10}$	-2.002303776843	$\times 10^{-10}$
19	-8.006218039652	$\times 10^{-11}$	0	
20	2.829596711623	$\times 10^{-11}$	2.143098365950	$\times 10^{-11}$

Table 2: Renormalized perturbation expansion for the Hamiltonian eq. (12) corresponding to the variable $(1-\kappa)$, for $\omega = 1$, without the factor $[\lambda/(1-\kappa)]^{1/2}$.

n	$E_n^{(r)}$	n	$E_n^{(r)}$	n	$E_n^{(r)}$
0	$1.15626707 \times 10^{+0}$	30	$-6.17135480 \times 10^{-4}$	60	$-5.42186148 \times 10^{-5}$
1	$-4.62506829 \times 10^{-1}$	31	$-8.97409509 \times 10^{-4}$	61	$-1.39700206 \times 10^{-5}$
2	$5.79387957 \times 10^{-2}$	32	$-1.05471829 \times 10^{-3}$	62	$2.19313729 \times 10^{-5}$
3	$8.79538466 \times 10^{-2}$	33	$-1.09662861 \times 10^{-3}$	63	$5.12970044 \times 10^{-5}$
4	$7.66512627 \times 10^{-2}$	34	$-1.03884674 \times 10^{-3}$	64	$7.27412089 \times 10^{-5}$
5	$5.72369534 \times 10^{-2}$	35	$-9.02533600 \times 10^{-4}$	65	$8.56777888 \times 10^{-5}$
6	$3.81037038 \times 10^{-2}$	36	$-7.11680394 \times 10^{-4}$	66	$9.02559362 \times 10^{-5}$
7	$2.18853370 \times 10^{-2}$	37	$-4.90726498 \times 10^{-4}$	67	$8.72486994 \times 10^{-5}$
8	$9.29467198 \times 10^{-3}$	38	$-2.62552992 \times 10^{-4}$	68	$7.79089589 \times 10^{-5}$
9	$2.69486228 \times 10^{-4}$	39	$-4.69358453 \times 10^{-5}$	69	$6.38082904 \times 10^{-5}$
10	$-5.59041145 \times 10^{-3}$	40	$1.40504096 \times 10^{-4}$	70	$4.66732911 \times 10^{-5}$
11	$-8.82442831 \times 10^{-3}$	41	$2.88858970 \times 10^{-4}$	71	$2.82321647 \times 10^{-5}$
12	$-1.00084522 \times 10^{-2}$	42	$3.92038967 \times 10^{-4}$	72	$1.00818515 \times 10^{-5}$
13	$-9.69724568 \times 10^{-3}$	43	$4.48490339 \times 10^{-4}$	73	$-6.41694726 \times 10^{-6}$
14	$-8.39124475 \times 10^{-3}$	44	$4.60609691 \times 10^{-4}$	74	$-2.02126036 \times 10^{-5}$
15	$-6.51827989 \times 10^{-3}$	45	$4.33946792 \times 10^{-4}$	75	$-3.06020107 \times 10^{-5}$
16	$-4.42531487 \times 10^{-3}$	46	$3.76289089 \times 10^{-4}$	76	$-3.72380971 \times 10^{-5}$
17	$-2.37729832 \times 10^{-3}$	47	$2.96715278 \times 10^{-4}$	77	$-4.01090885 \times 10^{-5}$
18	$-5.61149517 \times 10^{-4}$	48	$2.04694055 \times 10^{-4}$	78	$-3.94948537 \times 10^{-5}$
19	$9.06633351 \times 10^{-4}$	49	$1.09289125 \times 10^{-4}$	79	$-3.59067012 \times 10^{-5}$
20	$1.96998543 \times 10^{-3}$	50	$1.85144590 \times 10^{-5}$	80	$-3.00174130 \times 10^{-5}$
21	$2.62196298 \times 10^{-3}$	51	$-6.11340629 \times 10^{-5}$	81	$-2.25881443 \times 10^{-5}$
22	$2.89300170 \times 10^{-3}$	52	$-1.24961453 \times 10^{-4}$	82	$-1.43981869 \times 10^{-5}$
23	$2.83948651 \times 10^{-3}$	53	$-1.70176902 \times 10^{-4}$	83	$-6.18258640 \times 10^{-6}$
24	$2.53321973 \times 10^{-3}$	54	$-1.95836070 \times 10^{-4}$	84	$1.41864677 \times 10^{-6}$
25	$2.05221967 \times 10^{-3}$	55	$-2.02650229 \times 10^{-4}$	85	$7.89736574 \times 10^{-6}$
26	$1.47313226 \times 10^{-3}$	56	$-1.92695992 \times 10^{-4}$	86	$1.28991617 \times 10^{-5}$
27	$8.65397089 \times 10^{-4}$	57	$-1.69061944 \times 10^{-4}$	87	$1.62305601 \times 10^{-5}$
28	$2.87184260 \times 10^{-4}$	58	$-1.35467988 \times 10^{-4}$	88	$1.78528984 \times 10^{-5}$
29	$-2.16988780 \times 10^{-4}$	59	$-9.58901614 \times 10^{-5}$	89	$1.78650877 \times 10^{-5}$

Table 3: Ground–state energy calculated by the renormalized series and by direct numerical integration of the Schrödinger equation for a wide range of values of the coupling constant λ.

λ	κ	E(perturbative)	E(numerical)
0.1	0.90747	1.15816123	1.15816123
1	0.46140	1.29175416	1.29175416
10	0.05256	3.16909562	3.16909616
100	0.00315	9.99996845	10.00006874