On the elementary Schrödinger bound states and their multiplets

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Abstract

The problem of the existence of elementary bound states is discussed. A - trivial - observation that every elementary wavefunction $\psi^{[i]}(r)$ is an exact bound state for an appropriate potential, $V(r) = V^{[i]}[\psi(r), r]$, is shown to lead to a very transparent form of the "quasi-exact" (QE) solvability condition $V^{[i]} = V^{[j]}$ for doublets and multiplets of the ψ's. In this sense the particular class of elementary ansätze

$$\psi^{[i]}(r) = r^{\lambda} polynomial(r^2) \times \exp[r^{\mu} polynomial(r^2)]$$

also defines the particular class of QE-solvable potentials. They have an elementary non-polynomial (rational) form, possibly also with a strongly singular - repulsive - core at the origin. We discuss the properties of these forces in detail.

1 Introduction

It is well known that Schrödinger equation for harmonic oscillators is exactly solvable. Their potential $V = V^{[0]}(r) = A\,r^2$ with $A > 0$ is a rather exceptional force. In accord with Newton [1] we may only list three other completely solvable radial Schrödinger equations

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{l(l+1)}{r^2} + V^{[i]}(r)\right] = E\psi(r), \quad l = 0, 1, \ldots . \tag{1}$$

The first one, with Coulombic $V^{[1]}(r) = B/r$ and $B < 0$, proves obtainable simply by the change of variables $r \longrightarrow r^{'} = r^2$ in the harmonic-oscillator eq. (1) [2]. In the second case the square-well potential function, $V^{[2]}(r) = Ar^{\infty}$, remains discontinuous. Finally Newton's last case, the solvable - repulsive - spike with $V^{[3]}(r) = A\,r^{-2}$ does not possess any bound states at all.

Weaker definitions of the exact solvability may be - and are currently being - accepted. Even the mere restriction of eq. (1) to a single partial wave ($l = l_{fixed}$) enables one to solve this equation already for a fairly broad class of the so called - Bargmann potentials [1].

Recently a strengthening tendency towards further weakening of the solvability concept may be noticed. Following the pioneering work of Singh et al [3] devoted to the potential

$$V(r) = Ar^2 + Br^4 + Cr^6, \quad C > 0 \tag{2}$$

many people started feeling happy even when a *single* solution of eq. (1) acquired an elementary form. The construction of these partial or "quasi-exact" (QE) solutions was extended by Magyari [4] to arbitrary polynomials.

During the later development the interest in polynomial QE forces was complemented by studies of the non-polynomial interaction

$$V(r) = Ar^2 + \frac{Br^2}{1 + Cr^2}, \quad C > 0,\ A > 0 \tag{3}$$

[5 – 11], of the strongly singular potential

$$V(r) = Ar^2 + Br^{-4} + Cr^{-6}, \quad C > 0,\ A > 0 \tag{4}$$

[12] and of the various more complicated descendants of these examples. Here we intend to propose a further extension or "natural closure" of this QE class of $V(r)$'s.

We shall start from a fixed, postulated structure of the ground-state $\psi(r)$'s. In the present paper we shall only use exponentials, powers and polynomials as elementary functions. Thus we take the wavefunctions to be of the form

$$\psi(r) = r^{\lambda} polynomial(r^2) \times \exp[r^{\mu} polynomial(r^2)]. \tag{5}$$

Arbitrary superpositions of the above examples (2), (3) and (4) will be shown to enter this QE-solvability scheme in a natural way (Sect. 2).

In Sect. 3, the formal challenge of constructing and/or proving the simultaneous existence of several elementary bound states $\psi = \psi^{[j]}(r)$, $j = 1, 2, \ldots, t$ for the same potential $V(r)$ will be analysed. Next, Sect. 4 will be devoted to the underlying algebraic considerations. Sect. 5 will complement it by a short discussion of the possible moves beyond the immanent QE-solvability limitations. Sect. 6 will provide a summary.

2 The concept of QE solvability

In the light of the various possible (e.g., perturbative [13], [14]) applications of the QE solutions and/or of their multiplets, we believe that their completion and general analysis remains a topical problem. An indirect indication of its importance may be documented by the multiple (independent) rediscoveries of, say, the QE results of Singh et al. [3] (cf, e.g., refs. [15-19]).

The initial encouragement of our present new QE study stemmed from a rather puzzling mixture of trivial and non-trivial features in virtually any QE solvable problem. In particular a strong motivation came from constructions of multiplets as described, e.g., in ref [12] on the singular eq. (4) or in refs. [9,20-22] devoted to the nonpolynomial anharmonicity (3).

2.1 The re-construction of a single potential

An explicit construction of a single elementary bound state becomes entirely trivial once we proceed in an 'inverse' manner [23] and postulate some (ie, arbitrary, not necessarily elementary) form of the exact bound state $\psi(r)$ [7, 24]. As a consequence our partially solvable forces will be obtainable easily from eq. (1) once we re-read it as an explicit formula for, or definition of, $V(r)$ itself, *viz.*

$$V(r) = E - \frac{l(l+1)}{r^2} + \frac{\psi''(r)}{\psi(r)}. \tag{6}$$

Obviously any imposed ground-state wavefunction $\psi = \psi_0^{[i]}(r)$ generates the corresponding potential $V = V^{[i]}(r)$ explicitly, via the simple re-arrangement (6) of eq. (1). In a phenomenological context the trivial inverse procedure is often used, e.g., in spin systems [23] or laser theory [25]. Of course most papers on partial or 'quasi-exact' solvability of eq. (6) deal with its less trivial aspects. Thus let us start with their short - and by far incomplete - review.

2.2 The possible applications of QE solvability

For the sake of brevity people often select polynomial potentials as the simplest models which may be partially solvable [4, 26]. Several formal aspects appeared as a motivation for these studies. We may recall, e.g., the presence of continued fractions which terminate [3] and/or the above-mentioned change of variables giving the Coulomb + linear + harmonic Fivel modification of eq. (2) [27].

Results of this type may be generalized in certain very natural ways [28]. Again a broad variability of the QE forces themselves (e.g., after the same change of variables or after a less straightforward transition to the non-central [29] or manybody [19] cases) is worth mentioning.

QE considerations are usually assisted by detailed analyses of some separate examples. Indeed, starting from the simplest sextic anharmonic example of Singh et al [3] and ending up, e.g., with quasi-Schrödinger equations in some curved spaces [18], the examples always clarify our understanding of the QE phenomenon. In this setting studies of the algebraic properties of the QE models [17] or of their supersymmetric background [30] or recurrent features [31] should be also recalled. QE examples play an important role in the analyses of $1/N$ expansions [32], in the approximative extrapolations [33] etc.

A methodically important extension of the QE horizon was provided by transition to non-polynomial forces of the form $V = polynomial/polynomial$ [5-7, 10], possibly also with singularities in $V(r)$ at finite values of r [11]. The well behaved forces of the latter type became very popular as a testing ground. A particular emphasis has been laid upon the simplest nontrivial non-polynomial anharmonicity, $x^2/(1+gx^2)$ [8-11] and/or its screened-Coulombic equivalent [34]. We may remind the reader of the pertinent interesting algebraic results and observations of Whitehead et al [35] who proposed to treat certain sets of solutions as multiplets of Sturmians. Adhikari et al [36] re-emphasized the relevance of rational examples in supersymmetric quantum mechanics.

In a purely computational context new features of the non-polynomial case were noticed in refs. [37] and [38]. Whenever these forces become singular, we immediately loose the firm background of standard theory [39]. This is the reason why the singular case has separately been studied by several authors [12, 40]. Vice versa we may only apply the relevant and well established Floquet mathematical theory [39] to the simplest (e.g., inverse-sextic) singular anharmonicities [41].

In a way an ultimate methodical QE extreme is represented by the forces which are non-polynomial (rational) and strongly singular (repulsive) in the origin at the same time. A particular attention has to be paid to the extremely natural character of this 'last-step' generalization which seems to have been, rather undeservedly, neglected in the literature up to now. It preserves all the merits of feasibility of the simpler QE potentials.

In this paper we shall study, within these extended QE limits as specified by eq. (5) above, the general problem of existence (and also of the construction) of multiplets - mostly doublets - of bound states. We shall put emphasis upon the specification and representations of some underlying algebra and symmetries. Marginal attention will also be paid to the role that QE solutions may have in the various numerical algorithms and standard as well as new approximation schemes.

3 The QE constructions

If one solves the radial Schrödinger equation (1) numerically, one may encounter unexpected difficulties indeed. For example one of the standard methods for numerical computation based on the Prüfer transform [42] in the NAG Library [43] might fail for singular potentials (cf. Ramatlhape et al [44] who reported that the more robust algorithm of Gorringe and Leach [45] was to be preferred). Thus precisely the above-mentioned exact solutions of the singular and fractional type are needed there for testing. Of course, the complexity of their explicit construction may vary.

3.1 Trivial case - the ground-state-generated QE potentials

Let us postulate each exceptional eigenstate $\psi^{[i]}(r) = \psi(r)$ of a potential $V^{[i]}(r)$ in the form (5) or, better,

$$\psi(r) = D(r) \exp[-F(r)], \tag{7}$$

with $D(r)$ merely polynomial

$$D(r) = \sum_{n=0}^{N} d_n r^{2n} \tag{8}$$

and a slightly more complicated exponent

$$F(r) = -\lambda \log r + \sum_{m=0}^{M} f_m r^{2m+\mu}. \tag{9}$$

The simplest illustration may be taken over from Singh et al [3]. With trivial $D = 1$ and the simplest nontrivial $F(r) = \alpha r^2 + \beta r^4$ inserted in eq. (6), we recover their sextic polynomial potential (2) immediately. Similarly, in accord with Flessas [5], we may choose trivial $F(r) = \alpha r^2$ and nontrivial $D(r) = 1 + gr^2$ and get the second 'next-to-solvable' potential of eq. (3). We also obtain the strongly singular force (4) from the trivial $D = 1$ and nontrivial $F(r) = \alpha r^2 + \beta r^{-2}$. Thus we may expect that the next-to-the-simplest examples will result from a combined choice of nontrivial D's and F's.

In general, we may rewrite eq. (6) simply as an explicit formula or prescription

$$\begin{aligned} V(r) - E + \frac{l(l+1)}{r^2} &= \frac{\psi''(r)}{\psi(r)} \\ \Leftrightarrow W(r) - E &= F'^2 - F'' + \frac{(D'' - 2D'F')}{D} \end{aligned} \tag{10}$$

for the 'unknown' potential. Thus let us summarize the straightforward overall strategy which applies in the case of a single ground-state input ψ. We postulate an elementary formula (7) – (9) for ψ and get the explicit definition eq. (10) of the corresponding 'partially solvable' potential.

By way of further illustration let us fix nontrivial $N = 1$ and, say, $M = 2$ in eq. (9). Then, with the simplest negative $\mu = -2$, we have

$$D(r) = 1 + gr^2, \quad F(r) = ar^2 + b\log r + cr^{-2}, \quad a, c, g > 0 \tag{11}$$

after an inessential change in notation. By immediate evaluation, we again arrive at the specification of the underlying QE solvable potential

$$\begin{aligned} W(r) - E &= 4a^2r^2 + (4ab - 2a) + (b + b^2 - 8ac)/r^2 - (4b + 6)\frac{c}{r^4} \\ &+ 4\frac{c^2}{r^6} + \frac{8cgr^{-2} + (2g - 4bg) - 8agr^2}{1 + gr^2}. \end{aligned} \tag{12}$$

Various potentials discussed in the older literature may be obtained at the special values of the couplings from eq. (12), e.g., the rational eq. (3) results from the choice of $c = 0$, the multinomial singular eq. (4) is obtained at $g = 0$, and the polynomial - harmonic oscillator - comes out for trivial $g = c = b = 0$. We may again emphasize that the requirement of regularity $c = 0$ as usually used in the literature does not imply any simplification.

3.2 The QE multiplets

Now that we have guaranteed, in the trivial manner, the existence of 'the simplest' ground state of the form (7) and (11), let us consider a search for the second bound state solution of the same Schrödinger eq. (1) with the potential (12). In general, the assumption of elementarity of this second bound state $\hat{\psi}$ becomes much less trivial.

Obviously we must require that the new, 'hatted' elementary factors are of the form

$$\hat{D} = D\,P, \quad \hat{F} = F \tag{13}$$

with a polynomial P. The shift of energy and the coincidence of potentials

$$\hat{E} = E + \varepsilon, \quad \hat{W}(r) = W(r) \tag{14}$$

form the further conditions of existence of our doublet or, in general, of the several hatted components of a multiplet.

Insertion into eq. (10) gives immediately a simpler formulation

$$\varepsilon = \frac{D'' - 2D'F'}{D} - \frac{\hat{D}'' - 2\hat{D}'F'}{\hat{D}} \tag{15}$$

of our algebraic problem. We may also rewrite it as our final formula

$$\varepsilon = -\frac{2D'P'}{DP} + \frac{2P'F' - P''}{P} \tag{16}$$

which must be satisfied by the new 'unknown' wavefunction or, in the present notation, by the polynomial $P = P(r)$.

3.3 The simplest doublet - the constructive proof of existence

We take $D(r)$ and $F(r)$ as given in (11), denote

$$P = P(r) = 1 + fr^2 + hr^4 + zr^6, \tag{17}$$

and try to satisfy eq. (16). In the first step we notice that the right hand side of eq. (16) becomes singular in the limit $r \to 0$. Thus, as long as the left hand side - the energy difference ε - must remain r-independent and finite, we have to fix the value of the coupling f by the condition

$$f = 0. \tag{18}$$

As an immediate counterpart to the latter postulate, let us also imagine that the limiting transition $r \to \infty$ simplifies eq. (16) considerably. The second formula

$$\varepsilon = 24a \tag{19}$$

follows then simply from the leading-order part of this limit in eq. (16).

In the third step we notice that the right hand side function of eq. (16) also grows indefinitely near the (pair of) roots of $D(r)$. The disappearance of such a spurious pole necessitates that the polynomial D divides the product of polynomials $D'P'$ so that we have to fix

$$z = \tfrac{2}{3}gh. \tag{20}$$

We are left with the sufficiently simple form of eq. (16). It already admits a comparatively quick reduction to the almost elementary set of equations

$$a = -\tfrac{2}{3}hc, \quad 2b = 3 + 4cg, \quad 2hc + 6cg^2 = 9g \tag{21}$$

from which we have, in conjunction with the requirements of normalizability and regularity, the final specification of the (unique) solution

$$h = \frac{-3a}{2c} < 0, \quad z = \frac{-ga}{c} < 0, \quad b = 2gc + \tfrac{2}{3} > 0, \quad g = \frac{1}{4c}[3 + (9 + 8ac)^{1/2}] > 0. \tag{22}$$

An important observation concerns the nodal zeros of our new state ψ. As long as both h and z are negative, our second wavefunction may and must have just one nodal zero and, therefore, it has to be interpreted as the first excited state of our system.

In summary our construction specifies the couplings compatible with the current normalizability and regularity requirements. At least one QE doublet of bound states exists. We notice that this doublet degenerates to the example (3) for $c = 0$.

4 The algebraic background of QE solvability

Let us assume the existence of the second QE solution with $\bar{D}$ replacing D and $E+\varepsilon$ replacing E. Then, eq. (15) can be written in the form

$$(D\bar{D}' - D'\bar{D})' - 2F'(D\bar{D}' - D'D) + \varepsilon D\bar{D} = 0. \tag{23}$$

We may also choose any triplet of integers I, J and K as the respective highest (finite) powers of r in F, D and $\bar{D}$,

$$F = \sum^{I} f_i r^i, \quad D = \sum^{J} d_j r^j, \quad \bar{D} = \sum^{K} \bar{d}_k r^k. \tag{24}$$

In this - slightly modified - notation, eq. (23) becomes

$$\sum_j \sum_k \left[(j+k-1)(k-j)d_j\bar{d}_k r^{j+k-2}\right] - 2\sum_i \sum_j \sum_k \left[if_i(k-j)d_j\bar{d}_k r^{i+j+k-2}\right]$$
$$+ \varepsilon \sum_j \sum_k d_j\bar{d}_k r^{j+k} = 0 \tag{25}$$

in which the highest powers of r are $J+K-2$, $I+J+K-2$ and $J+K$ in the first, second and third sums respectively.

4.1 The nonlinear coupled algebraic equations

In the case that $I > 2$ the terms in the second sum in eq. (25) must self-cancel for powers greater than $J+K$. A simple calculation shows that $J = K$ and that, if D and $\bar{D}$ are normalized to $d_J = 1 = \bar{d}_K$, the coefficients $d_{J-i} = \bar{d}_{K-1}$ etc until the other terms in (25) are to be considered. If I is sufficiently large, all of the coefficients $d_j = \bar{d}_k$ and so $\varepsilon = 0$, ie, there is only one state for the potential given by eq. (10).

If I is not too large a number, there still remains some freedom in eq. (25) which is now reduced to a system of $2K+1$ equations in the $K+3-I$ undetermined coefficients $\bar{d}_k$ and the parameter ε. As such it is a constrained eigenvalue problem. Although this feature has not explicitly been noted before, we do not labour it, but rather consider it an alternative approach to determining the set of polynomials D which satisfy eq. (10).

As an elementary example we take

$$F = ar^4 + br^2, \quad D = c + r^2, \quad \bar{D} = d + r^2 \tag{26}$$

(by implication W is even and so $\bar{D}$ is chosen as even) which, when substituted into (25), leads to the three nontrivial equations (the other two are zero equals zero)

$$\begin{aligned} \varepsilon - 16a(c-d) &= 0 \\ \varepsilon(c+d) - 8b(c-d) &= 0 \\ \varepsilon cd + 2(c-d) &= 0. \end{aligned} \tag{27}$$

The solutions of (27) are

$$\begin{aligned} c &= [b \pm \sqrt{b^2 + 2a}]/4a \\ d &= [b \pm \sqrt{b^2 + 2a}]/4a \\ \varepsilon &= \pm 8\sqrt{b^2 + 2a} \end{aligned}$$

which are consistent with the well known results for the even sextic potential.

The case $I = 2$ is typical of the case when a harmonic oscillator is perturbed by a potential which contains negative powers or is a rational function, the $\lambda x^2/(1 + gx^2)$ pertubation being the best studied example. In (25) the highest powers of r are $J + K - 2$, $J + K$ and $J + K$ in the first, second and third sums respectively. Equating coefficients of separate powers of r to zero starting from the $(J + K)$th we find that

$$\varepsilon = 4f_2(K - J) \tag{28}$$

which means that multiplets can only occur when $K \neq J$ which is the very opposite to the case $I > 2$. The next power gives

$$\frac{\bar{d}_{K-1}}{\bar{d}_K} = \frac{d_{J-1}}{d_J} + \frac{f_1}{f_2}(K - J) \tag{29}$$

and expressions for $\bar{d}_{K-n}$, $n = 2, \ldots$ are readily found by routine manipulation.

In the context of $I = 2$ the ansatz (13) may profitably be used as it has been observed [7, 9] that the denominator of a rational potential occurs in the wavefunction.

The formula (16) becomes, in an obvious notation,

$$\sum_j \sum_l l(l+2j-1) d_j p_l r^{j+l-2} - 2\sum_i i f_i \left[\sum_j \sum_l l d_j p_l\right] r^{i+j+l-2} + \varepsilon \sum_j \sum_l d_j p_l r^{j+l} = 0, \tag{30}$$

with the highest power in P being r^L. The expression for ε is as in (28) with $K-J$ replaced by L. The next highest power of r yields

$$p_{L-1} = \frac{f_1 L}{2f_2} p_L \tag{31}$$

and further coefficients follow by means of routine arithmetic.

By way of example we take the well-known interaction

$$W = x^2 + \frac{\lambda x^2}{1+gx^2}, \quad x \in (-\infty, \infty) \tag{32}$$

in one dimension. The state

$$\psi = (1+gx^2)\exp(-\tfrac{1}{2}x^2) \tag{33}$$

exists provided the coupling constant λ is related to g via

$$\lambda = -2g(g+2). \tag{34}$$

The energy is given by

$$E = 1 - 2g. \tag{35}$$

The nature of the potential in (32) is such that the wavefunctions are either even or odd. We write

$$P = \sum p_i x^{2i+\delta}, \tag{36}$$

where $\delta = 0$ or 1 depending upon whether the state is even or odd. With this, $D = 1 + gx^2$ and $F = \frac{1}{2}x^2$, equation (30) becomes

$$\sum (2i+\delta)(2i + \delta - 1) p_i x^{2i+\delta-2} + \sum [\varepsilon + g(2i+\delta)(2i+\delta+3) - 2(2i+\delta)] p_i x^{2i+\delta} + g\sum [\varepsilon - 2(2i+\delta)] p_i x^{2i+\delta+2} = 0. \tag{37}$$

The coefficient matrix for the vector of p_i's is tridiagonal. The last term permits the separation of a finite part of the eigenvalue problem by setting $\varepsilon = 2(2I+\delta)$ and

$p_i = 0$, $i = I+1, \ldots$. The truncated part of (37) is now an eigenvalue problem on the parameter g.

We give some elementary results. When $\delta = 0$, for $I = 2$, $\varepsilon = 8$

$$P = p_0(1 - 4x^2 + 4x^4/7), \qquad g = -1/28 \tag{38}$$

and for $I = 3$, $\varepsilon = 12$

$$P = p_0(1 - 6x^2 + 4(g+1)x^4 - 8(g+1)x^6/27), \qquad g = \frac{-20 \pm \sqrt{148}}{63}. \tag{39}$$

It is noted that in both cases $g < 0$ so that we are treating the singular potential discussed in refs[10] and [36]. The original state is the second excited state and the subsequent states are the sixth and fourth or eighth (depending on the sign in g) respectively.

Let us now put $\delta = 1$. When $I = 1$, $\varepsilon = 6$, $g = -\frac{2}{3}$ we get $P = p_0 x(1 - 2x^2/9)$ which is the fifth excited state. When $I = 2$, $\varepsilon = 10$, $g = (-13 \pm \sqrt{133})/18$ and we have $P = p_0 x(1 - (2g+4)/3\, x^2 + (g+4)/15\, x^4)$. We may also note that, again, in both cases we are in the singular, $g < 0$, regime.

In full analogy with the above analysis of $r \to \infty$, we might parallel it by the similar discussion concerning the lowest powers of r and the equation (25) in the vicinity of zero $r = 0$. This is quite straightforward and will not be done here.

4.2 The underlying "quasi-dynamical" Lie algebras

It is well-known that many quasi-exactly solvable problems may be related to a finite dimensional representation of a Lie algebra. Turbiner [17] was the first to treat the problem of the sextic potential via the representation

$$J^+ = z^2\frac{\mathrm{d}}{\mathrm{d}z} - 2jz, \quad J^0 = z\frac{\mathrm{d}}{\mathrm{d}z} - j, \quad J^- = \frac{\mathrm{d}}{\mathrm{d}z}, \tag{40}$$

of $sl(2,R)$ where $z = x^2$ and $j(j+1)$ is the eigenvalue of the Casimir operator. Indeed the commutators of these operators give the standard $sl(2,R)$ relations

$$[J^+, J^-] = -2J^0, \quad [J^0, J^\pm] = \pm J^\pm. \tag{41}$$

One of the difficulties with this representation of the elements of $sl(2,R)$ is that one is limited to polynomial potentials of degree six. Recently de Souza Dutra and Boschi Filho [46] used a generalization of this representation to attain polynomial potentials of higher degree via the algebra $so(2,1)$ (isomorphic to $sl(2,R)$). The representation used was

$$\begin{aligned} T_1 &= \alpha_2 x^{2-j}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 + \alpha_1 x^{1-j}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) + \alpha_0 x^{-j} \\ T_2 &= -\frac{i}{j}x\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) - i\sigma \qquad T_3 \;=\; \alpha x^j \end{aligned} \tag{42}$$

which satisfies the $so(2,1)$ algebra

$$[T_1, T_2] = -iT_1; \quad [T_2, T_3] = -iT_3; \quad [T_1, T_3] = -iT_2 \tag{43}$$

provided only that

$$\sigma = \frac{1}{2j}\left[\frac{\alpha_1}{\alpha_2} + j - 1\right], \quad \lambda = -(2\alpha_2 j^2)^{-1}. \tag{44}$$

The constants α_1, α_2 and j are selected to match the different elements of the Hamiltonian.

The obvious question is whether the most general Hamiltonian may be given a representation in terms of the generators of the $sl(2,R)$ or $so(2,1)$ algebras. The work of de Souza Dutra and Boschi Filho provides a clue in this regard.

Their approach was to write

$$H = \sum a_k (T_3)^k T_1 + \sum b_l (T_3)^l T_2 + \sum c_m (T_3)^m \tag{45}$$

and, since T_3 was independent of $D := \mathrm{d}/\mathrm{d}z$, the third term permitted the introduction of an arbitrary polynomial potential. The ansätze of (42) and (45) can be naturally generalized to

$$\begin{aligned} T_1 &= f_0 + f_1 D + f_2 D^2 \\ T_2 &= g_0 + g_1 D \\ T_3 &= h_0 \end{aligned} \tag{46}$$

and

$$H = a(T_3)T_1 + b(T_3)T_2 + c(T_3), \tag{47}$$

where the coefficient functions are to be determined.

The primary constraint on the generators in (46) is that they satisfy (43) and hence the system of coupled equations

$$\begin{aligned}
-if_2 &= 2f_2g_1' \\
-if_1 &= f_1g_1' - f_1'g_1 + 2f_0g_0' + f_2g_1' \\
-if_0 &= -f_0'g_1 + f_1g_0' + f_2g_0' \\
-ig_0 &= f_0h_0' + f_2h_0' \\
-ig_1 &= 2f_2h_0' \\
-ih_0 &= g_1h_0'
\end{aligned} \tag{48}$$

must be solved. This leads to the solutions

$$\begin{aligned}
T_1 &= \frac{c+g_0^2}{h_0} + i\left(\frac{g_0}{h_0} - \frac{g_0'}{h_0'}\right) - \frac{1}{2}i\left(\frac{g_0}{h_0'} + i\frac{h_0h_0'}{h_0'^3}\right)D - \frac{h_0}{h_0'^2}D^2 \\
T_2 &= g_0 - \frac{ih_0}{h_0'}D \\
T_3 &= h_0,
\end{aligned} \tag{49}$$

where g_0 and h_0 are arbitrary functions and c is an arbitrary constant.

It might be wondered whether a more general representation would be possible. Suppose that we take the generators for the $sl(2,R)$ algebra to be

$$\begin{aligned}
J^+ &= a + bD + cD^2 \\
J^- &= d + eD + fD^2 \\
J^0 &= g + hD + iD^2\,,
\end{aligned} \tag{50}$$

where the functions a through i are to be determined through the relations

$$[J^+, J^-] = -2J^0, \quad [J^0, J^\pm] = \pm J^\pm.$$

After a calculation of tedious length we find that

$$J^+ = G_1$$

$$\begin{aligned} J^- &= L^2 G_1 + 2LG_2 + G_3 \\ J^0 &= LG_1 + G_2 \, , \end{aligned} \tag{51}$$

where

$$\begin{aligned} G_1 &= a + ruD + r^2 D^2 \\ G_2 &= \frac{1}{4}\left\{ (r' - u)\left(2LQ + \int \frac{1}{r}\right) - 1 \right\} - \frac{1}{2}\left(2LQ + \int \frac{1}{r}\right) D \\ G_3 &= \frac{1}{4}\left(2LQ + \int \frac{1}{r}\right)^2 . \end{aligned} \tag{52}$$

Here L and Q are arbitrary constants, r and u are arbitrary functions and the function a is the solution of the first order equation:

$$\begin{aligned} 0 &= \frac{1}{4} r^2 \left\{ (r' - u)\left(2LQ + \int \frac{1}{r}\right)\right\}'' + \frac{1}{4} ru \left\{ (r' - u)\left(2LQ + \int \frac{1}{r}\right)\right\}' \\ &\quad + \frac{1}{2} r \left(2LQ + \int \frac{1}{r}\right) a' + a \, . \end{aligned} \tag{53}$$

This can be integrated to give

$$a = K \left(2LQ + \int \frac{1}{r}\right)^{-2} + \frac{1}{4}(r' - u)^2 - \frac{1}{4}(r' - u)\left(2LQ + \int \frac{1}{r}\right)^{-1} - \frac{1}{2} r (r' - u)' \, ,$$

where K is an arbitrary constant. However, there is no generalisation as G_1, G_2 and G_3 are essentially the same as T_1, T_2 and T_3. In fact in terms of writing down a Hamiltonian operator as a combination of J's there is less freedom than with the T's.

In summary the analysis of the rational QE forces with a strong singularity in the origin remains very similar to its preceding simpler special cases. Now an important feature of the generalization may be seen in the increase of singularities in the underlying differential equation. Only some very general methodical considerations may be applied now to the construction of the non-QE solutions. This is very much of a challenge for applications.

5 A few analytic aspects of QE solvability

One of the most straightforward generalizations of the QE solutions may be based on a replacement of the polynomial (say, $D(r)$ (8)) by its $N \to \infty$ Taylor-series limit. In accord with the underlying theory of ordinary differential equations [39], every (i.e., normalizable as well as non-normalizable) solution $\psi(r)$ which pertains to the force $V(r)$ *regular in the origin* may be then expressed in the form of eq. (7). Thus, the new, $N \to \infty$ extended class of $D(r)$'s becomes already "too rich" - its restriction must be guaranteed by an independent imposition of the physical boundary conditions [47]. Nevertheless terminating solutions remain useful here as a non-numerical source of information.

5.1 QE solutions as points of degeneracy of the power-series method

Of course a limiting transition $N \to \infty$ in $D(r)$ (8) may (and often does [47]) violate the physical asymptotic boundary conditions. Hence any QE-like formal solution

$$\psi(r) = Taylor\ series \times an\ auxiliary\ exponential \tag{54}$$

need not represent a bound state in general. Vice versa there exists a number of examples [3,47] where it does specify the exact solution at the correct energy.

In the QE context, it is interesting to notice that the standard Frobenius power-series ansatz may happen to coincide with an exact (i.e., QE) solution if and only if its auxiliary exponential factor remains asymptotically equivalent to the physical WKB wavefunction,

$$\psi(r) = \psi_{WKB}(r) \sum_{n=0}^{N} h_n r^{n+l+1}. \tag{55}$$

In such a case the QE solutions may be obtained simply from the standard $N \to \infty$ Frobenius recurrences for the h's [39], complemented by the obvious conditions of termination at a finite dimension $M < \infty$,

$$\left(\begin{array}{cccccc} Q_{0,0} & Q_{0,1} & 0 & 0 & \ldots & 0 \\ Q_{1,0} & Q_{1,1} & Q_{1,2} & 0 & \ldots & 0 \\ & & & \ldots & & \\ Q_{M-1,0} & Q_{M-1,1} & & \ldots & & Q_{M-1,M} \\ Q_{M,0} & Q_{M,1} & & \ldots & & Q_{M,M} \\ Q_{M+1,0} & Q_{M+1,1} & & \ldots & & Q_{M+1,M} \\ & & & \ldots & & \end{array} \right) \left(\begin{array}{c} h_0^{[q]} \\ h_1^{[q]} \\ \vdots \\ h_M^{[q]} \end{array} \right) = 0. \qquad (56)$$

Here $M = M^{[q]}$ and the superscript q is introduced to run over all the members of the multiplet, $q = 1, 2, \ldots, t$.

Often the overcomplete set of linear algebraic QE eqs. (56) still remains finite and may be solved non-numerically (cf., e.g., [4], [7] or [12]). In the infinite-dimensional limit, $M \to \infty$, these equations may (but need not) remain solvable and their solutions may (though need not) remain physical. For more details on a rigorous analysis of this limiting transition (and on the related computational, so called "Hill-determinant", method), we refer, e.g., to Hautot's review [47].

5.2 Forces singular in the origin and doubly infinite Hill determinants

In the standard mathematical textbooks (e.g., [39]), the usage of the name of Hill determinants differs from ref. [47]. The name remains reserved to situations where the force $V(r)$ is strongly repulsive in the origin. In such a context the above limiting transition $N \to \infty$ must be complemented by the decrease of $\lambda \to -\infty$ in eq. (8). *Mutatis mutandis* most of the other statements of the preceding subsection remain valid again, and only the matrix Q in eq. (56) has to be replaced by its doubly infinite generalization

$$\begin{pmatrix} & & \cdots & \\ Q_{-L,0} & Q_{-L,1} & \cdots & Q_{-L,M} \\ & & \cdots & \\ Q_{-1,0} & Q_{-1,1} & \cdots & Q_{-1,M} \\ Q_{0,0} & Q_{0,1} & \cdots & Q_{0,M} \\ & & \cdots & \\ Q_{M,0} & Q_{M,1} & \cdots & Q_{M,M} \\ Q_{M+1,0} & Q_{M+1,1} & \cdots & Q_{M+1,M} \\ & & \cdots & \end{pmatrix} \begin{pmatrix} h_0^{[q]} \\ h_1^{[q]} \\ \vdots \\ h_M^{[q]} \end{pmatrix} = 0. \tag{57}$$

The new situation is of course not entirely identical. Let us only mention that the new generalized ansatz

$$\psi(r) = \mathit{Laurent\ series} \times \mathit{an\ auxiliary\ exponential} \tag{58}$$

need not suffice for representing bound states anymore. In fact we have to replace l in eq. (55) by (a pair of) free parameters so that only a pair of the QE-like ansätze (58) with $l = l_1$ and $l = l_2$ may - and must - be employed in general. More details may be found, e.g., in ref. [41].

5.3 QE solutions as starting points of perturbative considerations

In the standard (say, Rayleigh-Schrödinger) perturbation theory, people often combine numerical algorithms with perturbative considerations and the computations may start from a QE solution $\psi^{[0]}(r)$ and potential $V^{[0]}(r)$ in principle. Thus, whenever the latter force lies sufficiently close to the one under investigation, the use of QE states may be of importance. Now let us briefly mention a few other possible strategies.

In the simplest examples of the type (2), several new perturbation-like uses of the QE solutions may be found in the recent literature ([22], [33]). Here let us just briefly mention the most straightforward one. Whenever we use the Hill-determinant method (i.e., basically an infinite-dimensional generalization of eq. (56 [47]), we may use an appropriate q-th QE state as an approximation [14].

Whenever we admit a (complex) pole in $V(r)$, we cannot use the above simple power-series methods at all - the rational functions $V(r)$ and their QE solutions $\psi(r)$ are better understood in a variational-like setting [35]. Then we may assume that a bound state in question is written as a finite sum

$$| \psi \rangle = \sum_n \, | n \rangle \, h_n, \tag{59}$$

in some complete - not necessarily orthonormalized - set of states $|n\rangle$. In the spirit of the above-mentioned - now inapplicable - Hill-determinant constructions, we may even use a WKB-inspired choice of the type

$$\langle r|n\rangle = D(r) \; r^n \exp[-F(r)]. \tag{60}$$

More details may be found in ref. [48].

6 Summary

An appeal of our new QE solvable examples lies in their methodical aspects. They may combine a strong singularity in the origin with the presence of poles of $V(r)$ in the complex plane so that any other non-numerical way of constructing bound states may represent a real challenge formally.

For polynomial potentials we may treat QE solvability as a certain degeneracy of a more universal Hill-determinant method. Polynomial plus singular interactions permit an application of Floquet theory and Laurent (= doubly-infinite) power-series expansions in the similar spirit [41]. Unfortunately the more complicated forces (and especially those with singularities in the complex plane) cannot be treated in such a spirit so that a common background of the QE phenomenon is not clear and unique.

Here we proposed a possible QE-classification scheme based on the ψ–initiated reconstruction of the potential. This approach proves efficient even in the case of QE multiplets. An unexpected simplicity of the related algebra has been demonstrated on a number of examples here. Several particular bound-state solutions with elementary structure seem tractable by the common and transparent formalism.

The presence of features not excluded in the present QE examples (and, especially, poles lying near the real axis) usually restricts the use and/or convergence of the variational methods. Thus all the possible QE solvable forces may find applications in the computational (as well as perturbative, etc.) approximation techniques.

Originally our present study was initiated by an optimistic expectation that an applicability of standard techniques and/or re-interpretations of the elementary QE bound states will adhere to some suitable Lie-algebra or SUSY representations. In this context our present results remain unsatisfactory and incomplete. Nevertheless we hope that we shall be able to cover more of these algebraic aspects of the QE phenomenon in the not too distant future.

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