ELEMENTARY DOUBLETS OF BOUND STATES OF THE RADIAL DIRAC EQUATION

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Abstract

For non-relativistic Schrödinger equations the lowering of their degree by substitution $\Psi(r) \to F(r) = \Psi'(r)/\Psi(r)$ is known to facilitate our understanding and use of their (incomplete, so called quasi-exact) solvability. We show that and how the radial Dirac relativistic equation may quasi-exactly be solved in similar spirit.

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1 Introduction

The two-component radial Dirac equation with energy E, mass M and centrifugal term $U(r) = \kappa/r$ [1] reads

$$\left[\begin{array}{cc} \partial_r - U(r) & M + W(r) - E - V(r) \\ M + W(r) + E + V(r) & \partial_r + U(r) \end{array} \right] \left[\begin{array}{c} f(r) \\ g(r) \end{array} \right] = 0. \qquad (1)$$

In a way similar to non-relativistic Schrödinger equation it proves exactly solvable for the electrostatic field of hydrogen atom $V(r) = \alpha/r$ accompanied, if necessary, by the auxiliary or external Lorentz scalar force $W(r) = \beta/r$ and by the possible central magnetic monopole charge Q in $\kappa = \pm\sqrt{(\ell+1)(\ell+1+2|Q|)}$ where $\ell = 0, 1, \ldots$. Marginally, let us note that $\kappa = \ell + 1 = 0$ is also admitted whenever $\beta^2 > \alpha^2$ and $Q \neq 0$ (cf. ref. [2] for details).

Recently, Brihaye and Kosinski [3] conjectured that a formal parallel between relativistic and non-relativistic quantum mechanics may be extended to many other models. Explicitly, they have demonstrated that the perturbation of hydrogen atom by the linear relativistic force $W(r) \sim \omega\, r$ not only resembles its non-relativistic Coulomb plus linear plus quadratic analogue but also shares the *incomplete*, so called quasi-exact (QE) solvability with it. At certain exceptional energies and couplings, elementary bound states were obtained by non-numerical means in a way which complements the 25 years old non-relativistic result by A. Hautot [4].

In our present letter we intend to proceed one step further. Having in mind a deep non-relativistic connection between QE solvability and Riccati-Schrödinger equations [5], we shall formulate a parallel relativistic "order-lowering" idea and implement it in the technically slightly more difficult context of relativistic eq. (1). This will enable us to show that, in particular, the algebraic-equation approach of Brihaye and Kosinski just picks up a very specific portion of a much larger class of all the QE solvable Dirac equations.

2 New QE solutions: Explicit method

Once we represent a wavefunction in non-relativistic quantum mechanics as an integral $\Psi_0(r) = \exp \int_{r_{ini}}^{r} F(\xi)d\xi$ we get $\Psi_0''(r,\ell)/\Psi_0(r,\ell) = [F(r)]^2 + \partial_r F(r)$. This converts the radial differential Schrödinger equation to an equivalent first-order form $V(r) = E_0 - \ell(\ell+1)/r^2 + [F(r)]^2 + \partial_r F(r)$. Such a Riccati-type re-arranged equation is nonlinear but may be re-interpreted as an explicit closed definition of a partially solvable potential in terms of any of its "tentative" wavefunctions.

The latter point of view plays an important role in the understanding of non-relativistic QE systems [5]. *Mutatis mutandis*, the relativistic pair of equations (1) may also define the QE solvable pairs of potentials $W(r)$ and $V(r)$. Indeed, assuming that the latter forces are responsible for the existence of any particular elementary wavefunction with components $f_0(r)$ and $g_0(r)$ at a particular mass M_0 and energy E_0 we may write

$$W(r) = -M_0 - \frac{1}{2}\left[\frac{1}{g_0}\left(\partial_r - U\right) f_0 + \frac{1}{f_0}\left(\partial_r + U\right) g_0\right],$$

$$V(r) = -E_0 + \frac{1}{2}\left[\frac{1}{g_0}\left(\partial_r - U\right) f_0 - \frac{1}{f_0}\left(\partial_r + U\right) g_0\right]. \qquad (2)$$

On the basis of experience with non-relativistic models it is not too surprising that the mere correct threshold and asymptotic behaviour in tentative $f_0(r) = p\, r^\mu \exp(-\lambda\, r)$ and $g_0(r) = q\, r^\mu \exp(-\lambda\, r)$ with normalization constants p and q already leads to the exactly and, incidentally, completely solvable model as mentioned above (cf. also [3]). The next tentative elementary choice could mimick an unphysical singularity at negative $r_u = -1/h < 0$,

$$f_0(r) = p\, r^\mu (1 + h\, r) \exp(-\lambda\, r), \qquad g_0(r) = q\, r^\mu (1 + h\, r) \exp(-\lambda\, r). \qquad (3)$$

In terms of the same parameters $\varepsilon = \pm 1$ and $t \in (-\infty, \infty)$ in the input ratio of norms $p/q = \varepsilon \exp t$ we get the same formula for energy $E = -\varepsilon\, \lambda \sinh t$ and mass $M = \varepsilon\, \lambda \cosh t$ as above. Also both the Coulombic couplings remain the same,

$$\left(\begin{array}{c} \beta \\ \alpha \end{array}\right) = \left(\begin{array}{cc} -\cosh t & \sinh t \\ \sinh t & -\cosh t \end{array}\right)\left(\begin{array}{c} \varepsilon\, \mu \\ \varepsilon\, \kappa \end{array}\right). \qquad (4)$$

The only change emerges as a screening which enters the new and, by definition, QE solvable potentials

$$V(r) = \frac{\alpha}{r} + \frac{\alpha_s}{1+h\,r}, \quad W(r) = \frac{\beta}{r} + \frac{\beta_s}{1+h\,r}, \quad \alpha_s = \varepsilon\, h\, \sinh t, \quad \beta_s = -\varepsilon\, h\, \cosh t. \quad (5)$$

Further states in such a model may be sought and studied, by the explicit algebraic method of ref. [3], in full analogy with semi-relativistic and non-relativistic QE solvable screened Coulomb potentials [6].

3 QE solutions: Implicit method

Our new QE model (5) looks particularly simple after transition to the elementary integral representation of wavefunctions

$$f(r) = e^{\int_{r_{ini}}^{r} F(\xi)d\xi}, \qquad g(r) = e^{\int_{r_{ini}}^{r} G(\xi)d\xi}. \quad (6)$$

In non-relativistic setting, similar re-arrangement proved useful in computations [7] as well as in the so called supersymmetric transformations of Hamiltonians [8]. Also here, our postulate (6) will lead to simplifications. Thus, with abbreviations $Y(r) = [F(r)+G(r)]/2$ and $Z(r) = [F(r)-G(r)]/2$, one of the components of Dirac equation lowers its order and becomes purely algebraic,

$$[E+V(r)]^2 + [Y(r)]^2 = [M+W(r)]^2 + [U(r)-Z(r)]^2. \quad (7)$$

This enables us to parametrize, say,

$$E+V(r) = R(r)\cos A(r), \qquad M+W(r) = R(r)\cos B(r), \quad (8)$$

$$Y(r) = R(r)\sin A(r), \qquad U(r)-Z(r) = R(r)\sin B(r). \quad (9)$$

Such a transformation simplifies also the remaining Dirac equation

$$M + W(r) - E - V(r) + [Y(r)+Z(r)-U(r)]\, \exp 2\int_{r_{ini}}^{r} Z(\xi)d\xi = 0 \quad (10)$$

which, in terms of an abbreviation $C(r) = -[A(r)+B(r)]/2$, reads $\tan C(r) = \exp 2\int_{r_{ini}}^{r} Z(\xi)d\xi$ or, in differential form, $Z(r) = \partial_r C(r)/\sin 2C(r)$. After an insertion of such a definition of $Z(r)$ in the second item of eq. (9) we may eliminate the function $R(r)$ and are left with a set of the closed and compact simultaneous definitions of both the wavefunctions *and* potentials in terms of an arbitrary initial choice of the auxiliary but practically unrestricted pair of functions $A(r)$ and $B(r)$.

4 Re-parametrization

Our above construction of a QE solvable system exhibits still a certain similarity to its non-relativistic Riccati-like version. Unfortunately, the parallel is incomplete. In particular, we cannot derive potentials immediately from the wavefunctions since *both* of them enter our formulae together with their derivatives. At the same time, our implicit QE-type solution of Dirac eq. (1) still has to be made compatible with some overall physical requirements. Also its clearer physical interpretation is needed. For this purpose, let us make a further step. Recalling the standard Pauli matrices

$$\sigma_x = \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \qquad i\sigma_y = \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right), \qquad \sigma_z = \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \tag{11}$$

let us re-write eq. (1) as a manifestly real two-component problem

$$\{I\, \partial_r + [M + W(r)]\, \sigma_x - i[E + V(r)]\, \sigma_y - U(r)\, \sigma_z\}\ \psi(r) = 0 \tag{12}$$

and pre-multiply it by the transposed two-component real spinors $\sigma_x \psi(r)$ and $i\sigma_y \psi(r)$ from the left. The resulting pair of relations

$$\left[\begin{array}{c} Z(r) - U(r) \\ -Y(r) \end{array} \right] = \left[\begin{array}{cc} \cosh \Xi(r) & \sinh \Xi(r) \\ \sinh \Xi(r) & \cosh \Xi(r) \end{array} \right] \left[\begin{array}{c} E + V(r) \\ M + W(r) \end{array} \right] \tag{13}$$

with $\Xi(r) = 2 \int_{r_{ini}}^{r} Z(\xi) d\xi$ represents another, integral representation of our original Dirac bound-state problem. In it, the two-by-two matrix is easily invertible,

$$\left[\begin{array}{c} E + V(r) \\ M + W(r) \end{array} \right] = \left[\begin{array}{cc} \cosh \Xi(r) & -\sinh \Xi(r) \\ -\sinh \Xi(r) & \cosh \Xi(r) \end{array} \right] \left[\begin{array}{c} Z(r) - U(r) \\ -Y(r) \end{array} \right]. \tag{14}$$

This induces the simplified *ansatz*

$$E + V(r) = S(r) \sinh T(r), \qquad M + W(r) = S(r) \cosh T(r), \tag{15}$$

$$Z(r) = U(r) + S(r)\sinh[T(r) + \Xi(r)], \qquad Y(r) = -S(r)\cosh[T(r) + \Xi(r)] \quad (16)$$

which, in particular, parametrizes all the QE solutions by the independent input functions $S(r)$ and $T(r)$. As already mentioned above, they must only be subject to the appropriate physical boundary conditions. We may conclude that the *implicit* relativistic implementation of the idea of QE solvability is as straightforward as its *explicit* nonrelativistic predecessor.

5 Relativistic QE doublets

After any change of our above "parametrization" point of view, technical complications may re-emerge immediately. Even for the elementary and popular Coulomb + polynomial form of forces as suggested for further study of the QE solvability in ref. [3] we immediately imagine that a seemingly trivial guarantee of their compatibility with our parameterizations leads in fact to a quite difficult algebraic problem.

Another, mathematically easier *and* practically more important question is the possible existence and/or feasibility of constructions of the elementary QE multiplets. Indeed, in principle, after any choice of a QE wavefunction the resulting partially solvable potential may still remain compatible with *another* elementary bound state. Even in non-relativistic case such a physically useful requirement is mathematically non-trivial [9]. There, it may still be characterized by the comparatively transparent condition

$$\partial_r[F_1(r) - F_2(r)] + [F_1(r) - F_2(r)][(F_1(r) + F_2(r)] = E_2 - E_1. \qquad (17)$$

For the *same* QE potential (which was eliminated) this equation defines the superposition $F_s(r) = [F_1(r) + F_2(r)]/2$ in terms of the differences $F_d(r) = [F_1(r) - F_2(r)]/2$ and $\delta = E_2 - E_1$ [5]. This means that there exist very many nonrelativistic doublet partners $F_1 = F_s + F_d$ and $F_2 = F_s - F_d$ which are "numbered" by the choice of the virtually unconstrained functions $F_d(r)$.

In relativistic case, we have to proceed in similar vain. The elimination of the two energy-independent QE solvable potentials $V(r)$ and $W(r)$ from the two versions of eq. (14) will describe the difference between the two right hand sides as an $r-$independent spinor proportional to δ. We get a relativistic differential-equation counterpart to eq. (17) in terms of the auxiliary integrals $\alpha = \alpha(r) = \int^r (Z_2 - Z_1)$ and $\beta = \beta(r) = \int^r (Z_2 + Z_1)$ and their derivatives $Z_2(r) = \partial_r\alpha/2 + \partial_r\beta/2$ and $Z_1(r) = -\partial_r\alpha/2 + \partial_r\beta/2$,

$$\begin{pmatrix} Z_2 - U \\ -Y_2 \end{pmatrix} = \delta \begin{bmatrix} \cosh(\alpha+\beta) \\ \sinh(\alpha+\beta) \end{bmatrix} + \begin{pmatrix} \cosh 2\alpha & \sinh 2\alpha \\ \sinh 2\alpha & \cosh 2\alpha \end{pmatrix} \begin{pmatrix} Z_1 - U \\ -Y_1 \end{pmatrix}. \tag{18}$$

This formula may be read as an algebraic linear set of definitions of the two sums of exponents $Y_1(r)$ and $Y_2(r)$. The symmetry of our new equation with respect to the simultaneous double reflection $\beta(r) \leftrightarrow -\beta(r)$, $U(r) \leftrightarrow -U(r)$ and permutation $Y_1(r) \leftrightarrow Y_2(r)$ is one of the reasons why its closed solution is still unexpectedly compact,

$$Y_1(r) + Y_2(r) = \delta \frac{\cosh \beta(r)}{\sinh \alpha(r)} - \alpha'(r) \frac{\cosh \alpha(r)}{\sinh \alpha(r)}, \tag{19}$$

$$Y_1(r) - Y_2(r) = \delta \frac{\sinh \beta(r)}{\cosh \alpha(r)} + [\beta'(r) - 2U(r)] \frac{\sinh \alpha(r)}{\cosh \alpha(r)}. \tag{20}$$

We may summarize that our "parameters" $\alpha(r)$ and $\beta(r)$ remain practically arbitrary functions. In the relativistic Dirac case there exist infinitely many QE bound-state doublets as well. Thus, the well known *functional* freedom of the implicit doublet solutions of the non-relativistic Schrödinger QE equations is shared by our present Dirac relativistic QE construction.

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