Systematic search for $\mathcal{PT}$ symmetric potentials with real energy spectra

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Abstract

Changes of coordinates represent one of the most effective ways of deriving solvable potentials from ordinary differential equations for separate special functions. Here we relax the standard Hermiticity requirement and get an innovated construction which leads to unusual, complex potentials. Their energy spectrum is shown to stay real after a weakening of the Hermiticity of the Schrödinger equation to its mere invariance under the combined $\mathcal{P}$ (= parity) and $\mathcal{T}$ (= time reversal) symmetry. This ultimately results in richer bound-state spectra. Some of our new exactly solvable potentials generalize the current textbook models. Details are given for constructions based on the hypergeometric and confluent hypergeometric special functions.

I. INTRODUCTION

In the majority of quantum mechanical problems the Hamiltonian of the system is Hermitian, and this requirement guarantees that the energy eigenvalues are real. In some cases, however, the physical situation is such that the application of non-Hermitian Hamiltonians is justified. This happens, for example, for complex potentials used mainly in nuclear physics and accounting for the absorption of incident particles. In these models the discrete energy eigenvalues become complex in general. Recently another mechanism of weakening the Hermiticity requirement was also introduced in quantum mechanics. In this formalism of $\mathcal{PT}$ symmetric quantum mechanics [1] the Hamiltonian is required to be invariant under the simultaneous action of the $\mathcal{P}$ parity and the $\mathcal{T}$ time reversal operations. For one-dimensional potentials of nonrelativistic quantum mechanics this invariance requires $(V(-x))^* = V(x)$. Potentials exhibiting $\mathcal{PT}$ invariance are usually complex, nevertheless, their bound-state energy eigenvalues were often found to be real. In this analysis of $\mathcal{PT}$ invariant potentials various approaches have been applied such as Fourier transformation [2], semiclassical estimates, [3], numerical calculations [4], Sturm-Liouville-like theory [5], variational techniques [6] or perturbation methods [7].

Exact analytical solutions to some problems have also been given [8–18]. Most of the exactly solvable $\mathcal{PT}$ symmetric potentials have analogues in usual quantum mechanics. In some cases $\mathcal{PT}$ invariance is reached by simply setting the coupling constants of the odd potential terms to imaginary values. This was easy with potentials defined originally as one-dimensional problems on the full x axis [8–10]. In some other cases the coordinate x is shifted with an imaginary constant to $x - \mathrm{i}\epsilon$. One important aspect of this imaginary coordinate shift was that it cancelled the singularities typically appearing in some potentials at $x = 0$ (like the centrifugal barrier), and then these originally radial problems could be naturally extended to the full x axis [2,11,12]. For another class of potentials asymptotically deformed integration paths are defined to secure normalizability of the solutions [13–18].

It is not surprising that these exotic complex potentials had some unusual features. For example, bound (i.e. normalizable) states have been found at positive energies [18]. We note however, that a similar situation has been encountered for "conventional" complex potentials as well: the number of normalizable states was found to be larger than those having $\mathrm{Re}(E) < 0$ when the potential had imaginary component. See e.g. [19] for a simple analytical study for the Pöschl–Teller potential.

The aim of the present work is to study analytically solvable $\mathcal{PT}$ symmetric potentials by employing a method which has been used previously to derive exactly solvable potentials in a systematic way [20]. This method is based on transforming the Schrödinger equation into the second-order differential equation of a special function by appropriate variable transformations. In Ref. [20] it was found that this method leads to a natural classification scheme of solvable potentials; in particular the shape-invariant potentials [21] can all be found with it. It was also found that this approach can naturally be linked with the formalism of supersymmetric quantum mechanics [22] and some group theoretical methods [23].

In the present study we focus on the question of how the $\mathcal{PT}$ invariance requirement can be implemented for potentials with solutions related to the hypergeometric and confluent hypergeometric functions. With this systematic approach the exactly solvable $\mathcal{PT}$ symmetric potentials can be put into a more general framework, and one might also hope to find further solvable problems. It turns out that our method is rather suitable to interpret some aspects of $\mathcal{PT}$ symmetric potentials in a simple way. For example, the reality of the energy eigenvalues arises entirely naturally. Furthermore, potentials which are turned $\mathcal{PT}$ symmetric by an imaginary coordinate shift are all easily generated within the same approach.

In Sec. II we present the formalism of our method, and in Sec. III we apply it to potentials solved by hypergeometric and confluent hypergeometric functions. Finally, in Sec. IV we summarize the results.

II. THE BASIC FORMALISM

Following the discussion of Ref. [20], let us consider transformation of the Schrödinger equation

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + (E - V(x))\psi(x) = 0 \tag{2.1}$$

into the second-order differential equation of a special function $F(z)$. For this, we search for the solutions in the form

$$\psi(x) = f(x)F(z(x)) \ . \tag{2.2}$$

At the moment we do not specify the domain of definition for the coordinate x itself. Later on, in Sec. III we shall come back to this issue and its importance for $\mathcal{PT}$ symmetric problems.

Once we substitute Eq. (2.2) in our Schrödinger equation (2.1) we arrive at the ordinary differential equation of the special function $F(z)$

$$\frac{\mathrm{d}^2 F}{\mathrm{d}z^2} + Q(z)\frac{\mathrm{d}F}{\mathrm{d}z} + R(z)F(z) = 0 \tag{2.3}$$

where, by construction

$$\frac{z''}{(z')^2} + \frac{2f'}{z'f} = Q(z(x)) \tag{2.4}$$

and

$$\frac{f''}{(z')^2 f} + \frac{E - V(x)}{(z')^2} = R(z(x)) \ . \tag{2.5}$$

¿From these equations an explicit expression follows for $E - V(x)$:

$$E - V(x) = \frac{z'''(x)}{2z'(x)} - \frac{3}{4}\left(\frac{z''(x)}{z'(x)}\right)^2 + (z'(x))^2\left(R(z(x)) - \frac{1}{2}\frac{\mathrm{d}Q(z)}{\mathrm{d}z} - \frac{1}{4}Q^2(z(x))\right) \ . \tag{2.6}$$

Besides the functions $Q(z)$ and $R(z)$ defining the special function $F(z)$, this formula contains only the function representing a variable transformation, $z(x)$. This also applies to the solutions themselves. They can be written as

$$\psi(x) \sim (z'(x))^{-\frac{1}{2}} \exp\left(\frac{1}{2}\int^{z(x)} Q(z)\mathrm{d}z\right) F(z(x)) \ . \tag{2.7}$$

We are left with the task of finding such a functional form of $z(x)$ which takes our Schrödinger equation (2.6) into an exactly and *completely* solvable problem.

Of course, any randomly chosen $z(x)$ function satisfies the latter ambitious requirement for a particular potential $V(x)$ and energy E. We only cannot guarantee in general that any other physical solution of the same physical problem in question can be found in the same manner as well. In this perspective, a useful way of finding reasonable $z(x)$ functions has been proposed by Bhattacharjie and Sudarshan [24]. According to them, if there is a constant (E) on the left hand side of (2.6), then there must be one on the right hand side too. In Ref. [20] this fact was exploited, and a systematic list of potentials was compiled by identifying certain terms found on the right hand side of (2.6) with a constant C. This assignment leads to first-order differential equations for z of the type

$$\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2 \phi(z) = C \ , \tag{2.8}$$

where $\phi(z)$ is a function of z originating from $R(z)$ or $Q(z)$.

The general solution of the latter differential equation is given by formula

$$\int \phi^{1/2}(z)\mathrm{d}z = C^{1/2}x + \delta \ . \tag{2.9}$$

This defines an implicit function $x(z)$ and, in many cases of practical interest, also the explicit $z(x)$ function we need [20].

In the standard Hermitian context of the latter paper one considers mainly $\delta = 0$, in order to set $z(0) = 0$. The general $\delta \neq 0$ choice will be considered here instead. It corresponds to a shift of the coordinate and reflects a trivial and also rarely relevant transformation for potentials defined on the real x axis.

Its importance occurs, first of all, for all the radial problems in the Hermitian case. In what follows, we shall emphasize its enhanced relevance occurring also in the context of $\mathcal{PT}$ symmetric quantum mechanics. Mainly, we shall try to determine under what kind of conditions can one get a $\mathcal{PT}$ symmetric modification of the formalism of Ref. [20]. It turns out that the whole approach supplies a natural framework for these investigations and also helps to understand how and why $\mathcal{PT}$ symmetric complex potentials can have purely real spectra.

III. A SYSTEMATIC SEARCH FOR $\mathcal{PT}$ SYMMETRIC POTENTIALS

Here we apply the procedure outlined in Sec. II to potentials with bound-state solutions containing hypergeometric and confluent hypergeometric functions. Actually, these correspond to the rather general Natanzon [25] and Natanzon confluent [26] potential classes. In order to reduce the complexity of the formulae, we in fact, consider the Jacobi and the generalized Laguerre polynomials as the special function $F(z)$ in Eq. (2.2). These polynomials can be obtained from the $F(a, b; c; z)$ hypergeometric and the $F(a, c; z)$ confluent hypergeometric function by the $a = -n$ (or $b = -n$) substitution [27], and this special reduction represents an equivalent treatment with the general problem in all those cases where the physical bound-state solutions can be expressed with only one particular solution of the underlying second-order differential equation, the Schrödinger equation. For the sake of completeness, we show in the Appendix how the corresponding formulae are obtained for the hypergeometric and the confluent hypergeometric functions.

It was found in Ref. [20] that this method naturally lead to the identification of all known shape-invariant potentials [21], however, it is applicable to the systematic exploration of much wider potential classes, such as Natanzon potentials [25]. (These also include so-called implicit potentials, where the solutions $z(x)$ in (2.9) are obtained only in the implicit $x(z)$ form [28–32].) Now we adapt the formalism to the $\mathcal{PT}$ symmetric quantum mechanics. It turns out that as opposed to the original treatment, the integration constant δ now plays an important role; in fact it generally has to be chosen as a purely imaginary number to get $\mathcal{PT}$ symmetric problems. In what follows we chose the constant C to be real.

A. The Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$

In this case we have $Q(z) = (\beta-\alpha)(1-z^2)^{-1} - (\alpha+\beta+2)z(1-z^2)^{-1}$ while $R(z) = n(n+\alpha+\beta+1)(1-z^2)^{-1}$ and (2.6) has the form

$$E - V(x) = \frac{z'''(x)}{2z'(x)} - \frac{3}{4}\left(\frac{z''(x)}{z'(x)}\right)^2 + \frac{(z'(x))^2}{1-z^2(x)}\left(n+\frac{\alpha+\beta}{2}\right)\left(n+\frac{\alpha+\beta}{2}+1\right)$$
$$+\frac{(z'(x))^2}{(1-z^2(x))^2}\left[1-\left(\frac{\alpha+\beta}{2}\right)^2-\left(\frac{\alpha-\beta}{2}\right)^2\right] - \frac{2z(x)(z'(x))^2}{(1-z^2(x))^2}\left(\frac{\alpha+\beta}{2}\right)\left(\frac{\alpha-\beta}{2}\right). \quad (3.1)$$

Note that the parameters of the Jacobi polynomial, α and β appear everywhere only in the $(\alpha+\beta)/2$ and $(\alpha-\beta)/2$ combinations.

Let us consider first the PI case [20] defined by the differential equation $(z')^2(1-z^2)^{-1} = C$, which sets the third term on the right hand side of (3.1) to a constant. Rewriting the first two terms as the function of z and rearranging the equation we get

$$E - V(x) = C\left(n+\frac{\alpha+\beta+1}{2}\right)^2 + \frac{C}{1-z^2(x)}\left[\frac{1}{4}-\left(\frac{\alpha+\beta}{2}\right)^2-\left(\frac{\alpha-\beta}{2}\right)^2\right]$$
$$-\frac{2Cz(x)}{1-z^2(x)}\left(\frac{\alpha+\beta}{2}\right)\left(\frac{\alpha-\beta}{2}\right). \quad (3.2)$$

The $z(x)$ functions are the solutions of the differential equation defining the PI case, and their general form is given by the actual version of (2.9):

$$\int \frac{\mathrm{d}z}{(1-z^2)^{1/2}} = C^{1/2}x + \delta\ . \quad (3.3)$$

Depending on the nature of C (whether it is positive or negative) and that of z^2 (whether it is larger or smaller than 1), there are several solutions possible. In Ref. [20] we found five different cases labeled by $z(x) = \mathrm{i}\sinh(ax)$, $\cosh(ax)$, $\cos(ax)$, $\cos(2ax)$ and $\cosh(2ax)$ for $C = -a^2$, $-a^2$, a^2, $4a^2$ and $-4a^2$. The fourth and fifth of these cases yield essentially the same potentials as the third and the second cases, respectively, but the functional forms obtained for $V(x)$ are seemingly different and are handled traditionally as separate potentials. $z(x) = \sin(ax)$ is also a solution, but it gives the same potential as $z(x) = \cos(ax)$, only shifted with a π/a, therefore we did not consider it a separate solution in Ref. [20].

Let us now examine how these $z(x)$ functions behave under a $\mathcal{PT}$ transformation if we allow $\delta \neq 0$ in (3.3). The transformation properties of $z(x)$ also determine those of E and $V(x)$ in (3.2). It is easy to show that $\mathcal{PT}$ invariance of the potential cannot be reached in general if δ has a non-zero real component, because then the finite shift along the coordinate x renders $V(x)$ and its $\mathcal{PT}$ transformed version to essentially different forms. (There is an exception for those cases when $z(x)$ is a trigonometric function, because then the potentials are periodic. However, if we consider these potentials only within a single period, then $\mathcal{PT}$ invariance is lost for these special cases too.) If we set $\delta = \mathrm{i}\epsilon$, then the transformation properties of the $z(x)$ functions specified previously are the following:

$$
\begin{aligned}
&\mathcal{PT}: z(x)=\mathrm{i}\sinh(ax+\mathrm{i}\epsilon)\longrightarrow \tilde{z}(x)\equiv(\mathrm{i}\sinh(a(-x)+\mathrm{i}\epsilon))^{*}=\mathrm{i}\sinh(ax+\mathrm{i}\epsilon)=z(x)\ ,\\
&\mathcal{PT}: z(x)=\cosh(ax+\mathrm{i}\epsilon)\longrightarrow \tilde{z}(x)\equiv(\cosh(a(-x)+\mathrm{i}\epsilon))^{*}=\cosh(ax+\mathrm{i}\epsilon)=z(x)\ ,\\
&\mathcal{PT}: z(x)=\cos(ax+\mathrm{i}\epsilon)\longrightarrow \tilde{z}(x)\equiv(\cos(a(-x)+\mathrm{i}\epsilon))^{*}=\cos(ax+\mathrm{i}\epsilon)=z(x)\ ,\\
&\mathcal{PT}: z(x)=\sin(ax+\mathrm{i}\epsilon)\longrightarrow \tilde{z}(x)\equiv(\sin(a(-x)+\mathrm{i}\epsilon))^{*}=-\sin(ax+\mathrm{i}\epsilon)=-z(x)\ . \qquad (3.4)
\end{aligned}
$$

The first three cases have been considered previously in [20], while the fourth one has to be considered a new independent possibility if we generalize our study to $\mathcal{PT}$ symmetric quantum mechanics.

The $\mathcal{PT}$ transformed version of Eq. (3.2) is

$$
\begin{aligned}
\tilde{E}-\tilde{V}(x)\equiv E^{*}-(V(-x))^{*}=&\\
&=C\left(n+\frac{\alpha^{*}+\beta^{*}+1}{2}\right)^{2}+\frac{C}{1-\tilde{z}^{2}(x)}\left[\frac{1}{4}-\left(\frac{\alpha^{*}+\beta^{*}}{2}\right)^{2}-\left(\frac{\alpha^{*}-\beta^{*}}{2}\right)^{2}\right]\\
&-\frac{2C\tilde{z}(x)}{1-\tilde{z}^{2}(x)}\left(\frac{\alpha^{*}+\beta^{*}}{2}\right)\left(\frac{\alpha^{*}-\beta^{*}}{2}\right)\ . \qquad (3.5)
\end{aligned}
$$

(Remember that we chose C to be real.) It is clear from (3.2) and (3.5) that $\mathcal{PT}$ invariance of the potential is satisfied for the $\tilde{z}(x)=z(x)$ cases if $(\alpha^{*})^{2}=\alpha^{2}$ and $(\beta^{*})^{2}=\beta^{2}$ holds, i.e. for $\alpha^{*}=\pm\alpha$ and $\beta^{*}=\pm\beta$. This can happen if α and β are purely real or imaginary. Together with C and ϵ then there are 4 real parameters which define the potential shape.

When $\tilde{z}(x)=-z(x)$ holds, then the change of the sign in the last term in (3.5) has to be compensated with the appropriate choice of α and β. Requiring also $\mathcal{PT}$ invariance of the other potential term (which is an even function of $z(x)$) restricts the parameters to $(\alpha^{*})^{2}=\beta^{2}$, i.e. we get $\alpha^{*}=\pm\beta$. The $\alpha^{*}=\beta$ choice leads to $(\alpha+\beta)^{*}=\alpha+\beta$, and in this case the energy eigenvalues remain unchanged and are purely real. The number of real parameters are then, again 4, considering also C and ϵ. The $\alpha^{*}=-\beta$ choice also secures $\mathcal{PT}$ invariance of the potential, however, in this case the energy eigenvalues might become complex in principle, due to $(\alpha+\beta)^{*}=-(\alpha+\beta)$.

We listed the individual PI type potentials and the corresponding energy formulae in Table I, along with the conditions for $\mathcal{PT}$ invariance. For the sake of completeness we also display the $z(x)=\cosh(2ax+\mathrm{i}\epsilon)$ and $z(x)=\cos(2ax+\mathrm{i}\epsilon)$ options, which are not independent cases, rather they can be obtained from the $z(x)=\cosh(ax+\mathrm{i}\epsilon)$ and $z(x)=\cos(ax+\mathrm{i}\epsilon)$ cases by the $a\rightarrow 2a$ replacement, using also formulae connecting hyperbolic and trigonometric functions with similar functions having half the original arguments. We also included in Table I the $z(x)=\sin(ax+\mathrm{i}\epsilon)$ case, which did not appear in Ref. [20] as an independent problem, because $z(x)=\cos(ax)$ could be trivially obtained from $z(x)=\sin(ax)$ by a simple coordinate shift. As noted previously, real shifts of the coordinate are not compatible with $\mathcal{PT}$ invariance in general (e.g. $\delta=\mathrm{i}\epsilon$ is purely imaginary), therefore these two cases cannot be obtained from each other now, only if we define the potentials to be periodic.

The PI type potentials listed in Table I are complex in general, due to the $\mathrm{i}\epsilon$ constant. If we set $\epsilon=0$, the symmetric potential terms become real, while the odd ones turn purely imaginary. In fact, in this case the $z(x)=\cosh(ax+\mathrm{i}0)$ and the $z(x)=\cos(ax+\mathrm{i}0)$ potentials become fully real for any allowed α and β. The remaining two cases, $z(x)=\mathrm{i}\sinh(ax+\mathrm{i}0)$ and $z(x)=\sin(ax+\mathrm{i}0)$ present imaginary antisymmetric potential terms too.

There are also further special values of ϵ which deserve attention. It can be shown that the $\epsilon = 0$ version of the two hyperbolic PI type potentials can be obtained from the general $z(x) = \sinh(ax + \mathrm{i}\epsilon)$ and $z(x) = \cosh(ax + \mathrm{i}\epsilon)$ cases alike, when ϵ is set to $k\pi$ or $\pi/2 + k\pi$. This means that the two potentials can be continuously transformed into each other by carefully tuning ϵ.

According to Eq. (2.7), the solutions of the Schrödinger equation expressed in terms of a Jacobi polynomial have the form

$$\psi(x) \sim (1 - z(x))^{\frac{\alpha}{2}+\frac{1}{4}}(1 + z(x))^{\frac{\beta}{2}+\frac{1}{4}} P_n^{(\alpha,\beta)}(z(x)) \ . \tag{3.6}$$

The regularity of these wave functions can be controlled by appropriate relations for α, β and n, whenever $|z| \to \infty$ or $z = \pm 1$ can occur. For the case of $|z| \to \infty$ regular behaviour of $\psi(x)$ can be guaranteed by the prescription $n + [\mathrm{Re}(\alpha + \beta) + 1]/2 < 0$. This condition sets an upper limit for n: $n < -[\mathrm{Re}(\alpha + \beta) + 1]/2$. When $z = 1$ or $z = -1$ can occur, then the regularity of $\psi(x)$ requires $\mathrm{Re}(\alpha) > -1/2$ and $\mathrm{Re}(\beta) > -1/2$, respectively. Now let us see which of these conditions apply to the individual PI type potentials listed in Table I.

In the $z(x) = \mathrm{i}\sinh(ax + \mathrm{i}\epsilon)$ case only $|z| \to \infty$ has to be taken care of, for $x \to \pm\infty$. (We note that $z = \pm 1$ can also occur here if $\sin(\epsilon) = \pm 1$ holds, because in this case $z(x) \to \mp\cosh(ax)$. Since this special case corresponds to a particular example for the next PI type potential, we do not consider it here.) Then $n < -[\mathrm{Re}(\alpha + \beta) + 1]/2$ sets an upper limit for the number of bound states. This condition also means that there are no bound states for this potential if both α and β are imaginary. Special cases of this potential with $\epsilon = 0$ are mentioned in Ref. [8] ($\alpha = \frac{\lambda}{\mu} + 1$, $\beta = \frac{\lambda}{\mu} - 1$, $a = \mu$) and Ref. [9] ($\alpha = -b - A - \frac{1}{2}$, $\beta = b - A - \frac{1}{2}$, $a = 1$).

For $z(x) = \cosh(ax + \mathrm{i}\epsilon)$ the $n < -[\mathrm{Re}(\alpha + \beta) + 1]/2$ applies again, because $|z| \to \infty$ can occur. Now $z = \pm 1$ can also appear, if $\cos(\epsilon) = \pm 1$. In these cases a singularity appears in the potential at $x = 0$. (We note that this singularity also appears for the conventional version of this potential, which is considered a radial problem.) If we exclude these particular values of ϵ, then the potential becomes finite everywhere, and there are no further restrictions for the potential parameters. Similarly to the previous case, there are no bound states if both α and β are imaginary. The $\mathcal{PT}$ symmetric Pöschl–Teller potential discussed in Ref. [12] corresponds to this case, taking $\alpha = -A - 1/2$, $\beta = B - 1/2$, $C = -4$, $a = 1$ and using -2ϵ instead of ϵ.

In the trigonometric cases $z(x) = \cos(ax + \mathrm{i}\epsilon)$ and $z(x) = \sin(ax + \mathrm{i}\epsilon)$, $|z| \to \infty$ cannot occur, therefore no conditions limit the possible values of n. Furthermore, $z = \pm 1$ can also occur for $\epsilon = 0$ only, in which case these potentials have singularities at $ax = k\pi$, and $ax = (k + \frac{1}{2}\pi)$, respectively, similarly to the conventional versions of these problems. Then the $\mathrm{Re}(\alpha) > -1/2$ and $\mathrm{Re}(\beta) > -1/2$ conditions also have to be observed in both cases, and have to be combined with the other conditions for α and β required by $\mathcal{PT}$ symmetry.

The regularization of the potentials by eliminating their singularities with appropriate choices of ϵ relaxes the boundary conditions considerably. This means that in principle, the second independent solution of the Schrödinger equation (which is disqualified due to these boundary conditions for the conventional problems [33]) also becomes allowed. In fact, the general solution of the Schrödinger equation can then be written in terms of two hypergeometric functions as

$$\psi(x) \sim (1-y)^{\frac{1}{2}(a+b-c)+\frac{1}{4}} \left(C_1 y^{\frac{2c-1}{4}} F(a,b;c;y) + C_2 y^{\frac{3-2c}{4}} F(b-c+1,a-c+1;2-c;y) \right) \ , \tag{3.7}$$

where $y=(1-z(x))/2$. The particular solution (3.6) can be obtained from (3.7) by setting $C_2=0$, $a=-n$, $b=n+\alpha+\beta+1$ and $c=\alpha+1$, which reduces the remaining hypergeometric function to a Jacobi polynomial. (See Eq. 22.5.42 in Ref. [27] and also the Appendix for the general treatment of the hypergeometric function in terms of the formalism of Sec. II.) The second solution does not introduce anything essentially new, rather it just extends the allowed values of α (to $-\alpha$), according to the less strict boundary conditions.

Let us turn to the PII case [20] defined by the differential equation $(z')^2(1-z^2)^{-2}=C$, in which case the fourth term on the right hand side of (3.1) becomes a constant:

$$\begin{aligned} E - V(x) = &-C\left[\left(\frac{\alpha+\beta}{2}\right)^2 + \left(\frac{\alpha-\beta}{2}\right)^2\right] + C\left(n+\frac{\alpha+\beta}{2}\right)\left(n+\frac{\alpha+\beta}{2}+1\right)(1-z^2(x)) \\ &-2C\left(\frac{\alpha+\beta}{2}\right)\left(\frac{\alpha-\beta}{2}\right)z(x) \ . \end{aligned} \tag{3.8}$$

With a parameter transformation the n-dependence can be transferred to the constant (energy) term. The potential then can be written as

$$V(x) = -Cs(s+1)(1-z^2(x)) - 2C\Lambda z(x) \ , \tag{3.9}$$

where $s=n+\frac{(\alpha+\beta)}{2}$ or $s=-n-\frac{(\alpha+\beta)}{2}-1$ and $\Lambda=\frac{\alpha-\beta}{2}\frac{\alpha+\beta}{2}$. This gives $\alpha=s-n+\Lambda/(s-n)$, $\beta=s-n-\Lambda/(s-n)$, or $\alpha=-s-n-1-\Lambda/(s+n+1)$, $\beta=-s-n-1+\Lambda/(s+n+1)$. The energy eigenvalues are then given by $E=-C\left((s-n)^2+\frac{\Lambda^2}{(s-n)^2}\right)$ or $E=-C\left((s+n+1)^2+\frac{\Lambda^2}{(s+n+1)^2}\right)$. In order to simplify the formalism, in what follows we consider only the first set of the above relations: the second set can be obtained by the $s\rightarrow -s-1$ substitution.

The $z(x)$ functions are again supplied by the current version of (2.9):

$$\int \frac{\mathrm{d}z}{1-z^2} = C^{1/2}x + \delta \ . \tag{3.10}$$

In Ref. [20] the $\delta=0$ choice was made and three independent solutions were identified: $z=\tanh(ax)$, $\coth(ax)$ and $-\mathrm{i}\cot(ax)$ with $C=a^2$, a^2 and $-a^2$, respectively. One further solution, $z=\mathrm{i}\tan(ax)$ with $C=-a^2$ is essentially the same as the $-\mathrm{i}\cot(ax)$ case, therefore it was not discussed as a separate possibility.

Considering the $\mathcal{PT}$ symmetric case, we again find that the $\delta=\mathrm{i}\epsilon$ choice has to be made in order to reach $\mathcal{PT}$ invariance of the potentials. The transformation properties of the four possible $z(x)$ functions under the $\mathcal{PT}$ operation are the following:

$$\begin{aligned} \mathcal{PT}: z(x) = \tanh(ax+\mathrm{i}\epsilon) &\longrightarrow \tilde{z}(x) \equiv (\tanh(a(-x)+\mathrm{i}\epsilon))^* = -\tanh(ax+\mathrm{i}\epsilon) = -z(x) \ , \\ \mathcal{PT}: z(x) = \coth(ax+\mathrm{i}\epsilon) &\longrightarrow \tilde{z}(x) \equiv (\coth(a(-x)+\mathrm{i}\epsilon))^* = -\coth(ax+\mathrm{i}\epsilon) = -z(x) \ , \\ \mathcal{PT}: z(x) = -\mathrm{i}\cot(ax+\mathrm{i}\epsilon) &\longrightarrow \tilde{z}(x) \equiv (-\mathrm{i}\cot(a(-x)+\mathrm{i}\epsilon))^* = -\mathrm{i}\cot(ax+\mathrm{i}\epsilon) = z(x) \ , \\ \mathcal{PT}: z(x) = \mathrm{i}\tan(ax+\mathrm{i}\epsilon) &\longrightarrow \tilde{z}(x) \equiv (\mathrm{i}\tan(a(-x)+\mathrm{i}\epsilon))^* = \mathrm{i}\tan(ax+\mathrm{i}\epsilon) = z(x) \ . \end{aligned} \tag{3.11}$$

Similarly to the PI case, the last $z(x)$ function can be obtained from the other trigonometric one by using $\delta = -\pi/(2a) + \mathrm{i}\epsilon$ instead of $\delta = \mathrm{i}\epsilon$. However, we again considered it an independent case because finite real translations are not compatible with $\mathcal{PT}$ invariance in general. The $\mathcal{PT}$ transformed potential (3.9) becomes

$$\tilde{V}(x) = -Cs^*(s^*+1)(1-\tilde{z}^2(x)) - 2C\Lambda^*\tilde{z}(x) \tag{3.12}$$

and the corresponding energy eigenvalues are $\tilde{E} = -C\left((s^*-n)^2 + \frac{(\Lambda^*)^2}{(s^*-n)^2}\right)$. For the $\tilde{z}(x) = z(x)$ cases $\mathcal{PT}$ invariance is reached if $(s(s+1))^* = s(s+1)$ and $\Lambda^* = \Lambda$. This means that Λ has to be real, while s is either real, or $s = -\frac{1}{2} + \mathrm{i}\sigma$. In the first case α and β are both real, and the energy eigenvalues are also real. When $\tilde{z}(x) = -z(x)$, then for $\mathcal{PT}$ invariance we need $(s(s+1))^* = s(s+1)$ and $\Lambda^* = -\Lambda$. In this case Λ has to be imaginary, while s can have the same values as in the previous case. If s is real, then $\alpha^* = \beta$ holds, and the energy eigenvalues are real.

Table I contains the individual PII type potentials, the energy formulae, and the conditions for $\mathcal{PT}$ invariance. Similarly to the PI case, these potentials are also complex in general, and for $\epsilon = 0$ their symmetric terms become real, while their antisymmetric terms (the second term of all four potentials) turn purely imaginary. We again find that the $\epsilon = 0$ version of the two hyperbolic potentials can be obtained from the general $z(x) = \tanh(ax+\mathrm{i}\epsilon)$ and $z(x) = \coth(ax + \mathrm{i}\epsilon)$ cases alike, when ϵ is set to $k\pi$ or $\pi/2 + k\pi$, therefore the two potentials can be continuously transformed into each other. Yet another similarity with the PI case is that the singularities of the potentials appear only for special values of ϵ.

The solutions of the Schrödinger equation are now

$$\psi(x) \sim (1-z(x))^{\frac{\alpha}{2}}(1+z(x))^{\frac{\beta}{2}} P_n^{(\alpha,\beta)}(z(x)) \ . \tag{3.13}$$

The regularity of these wave functions has to be examined again for $|z| \to \infty$ and $z = \pm 1$. For $|z| \to \infty$ regularity requires $n < -[\mathrm{Re}(\alpha+\beta)]/2 = \mathrm{Re}(s) - n$, i.e. $n < \mathrm{Re}(s)/2$, while if $z = 1$ or $z = -1$ can happen, the conditions $\mathrm{Re}(\alpha) > 0$ and $\mathrm{Re}(\beta) > 0$ have to be observed. For the potentials listed in Table I these conditions mean the following.

In the $z(x) = \tanh(ax + \mathrm{i}\epsilon)$ case only $z = \pm 1$ can occur (for $x = \pm\infty$, irrespective of ϵ), and one needs $\mathrm{Re}(\alpha) > 0$ and $\mathrm{Re}(\beta) > 0$. If s is real, then these relations both lead to $s > n$, because $\mathrm{Re}(\Lambda/(s-n) = 0$, due to the imaginary value of Λ, prescribed by $\mathcal{PT}$ invariance. (We note that in the conventional case, when Λ can have real values, these conditions lead to more strict limits for n.) If s is complex: $s = -\frac{1}{2} + \mathrm{i}\sigma$, then the $\mathrm{Re}(\alpha+\beta) = 2\mathrm{Re}(s) - 2n = -1-2n > 0$ condition means that there can be no bound states in this potential.

For $z(x) = \coth(ax + \mathrm{i}\epsilon)$ the $\mathrm{Re}(s) > n$ condition applies again (for $z = \pm 1$, i.e. $x \to \pm\infty$), but $|z| \to \infty$ can also occur at $x = 0$, if $\epsilon = k\pi$. In this case a singularity appears, just like in the conventional version of this potential, which is therefore considered a radial problem. If we consider only the regularized (i.e. non-singular) $\mathcal{PT}$ symmetric potentials with $\epsilon \neq k\pi$, then we again find that bound states can appear only for real values of s. (In the conventional case with real parameters the boundary condition at $x = 0$ also requires a negative Λ with $-\Lambda > s^2$, which is reasonable, because only in this case can one have an attractive potential component which can support bound states.) This potential has already been considered as a $\mathcal{PT}$ symmetric problem [12] with the specific parameters $\Lambda = -B$, $a = 1$, $\epsilon = 0$ and $s = A - 1$ (or $s = -A$).

In the trigonometric cases $z(x) = -\mathrm{i}\cot(ax + \mathrm{i}\epsilon)$ and $z(x) = \mathrm{i}\tan(ax + \mathrm{i}\epsilon)$, $z = \pm 1$ cannot occur (irrespective of ϵ), and therefore there are no limits for n. $|z| \to \infty$ can also happen for $\epsilon = 0$; if these $\mathcal{PT}$ symmetric potentials are regularized by taking $\epsilon \neq 0$, then there are no conditions at all for the parameters following from the regularity of the wave functions. For $\epsilon = 0$ one obtains $\mathrm{Re}(s) < -\frac{1}{2}$, which basically sets the coupling parameter $s(s+1)$ of the x^{-2} type singular term to the repulsive or the "weakly attractive" regime, where physically acceptable solutions are possible.

Again, the relaxed boundary conditions (following from the cancellation of singularities) mean that the general solution of the Schrödinger equation should be considered, not just the particular solution in (3.13). This can be expressed in terms of two hypergeometric functions as

$$\psi(x) \sim (1-y)^{\frac{1}{2}(a+b-c)} \left(C_1 y^{\frac{c-1}{2}} F(a,b;c;y) + C_2 y^{\frac{1-c}{2}} F(b-c+1, a-c+1; 2-c; y) \right) , \tag{3.14}$$

with $y = (1 - z(x))/2$. The particular solution (3.13) can be obtained from (3.14) by setting $C_2 = 0$, $a = -n$, $b = n + \alpha + \beta + 1$ and $c = \alpha + 1$. The second solutions again, extend the allowed range of the α parameter to values $(-\alpha)$ which are not allowed when there are singularities for $y = 0$, i.e. $z(x) = 1$. This becomes relevant to the hyperbolic cases in the $x \to \pm\infty$ limit.

B. The generalized Laguerre polynomials $L_n^{(\alpha)}(z)$

For the generalized Laguerre polynomials $L_n^{(\alpha)}(z)$ [27] we have $Q(z) = -1 + (\alpha+1)/z$ and $R(z) = n/z$. The current form of (2.6) becomes

$$E - V(x) = \frac{z'''(x)}{2z'(x)} - \frac{3}{4}\left(\frac{z''(x)}{z'(x)}\right)^2 + \frac{(z'(x))^2}{z(x)}\left(n + \frac{\alpha+1}{2}\right) - \frac{(z'(x))^2}{4} - \frac{(z'(x))^2}{z^2(x)}\left(\frac{\alpha^2-1}{4}\right) \tag{3.15}$$

and the corresponding solutions, according to (2.7) are

$$\psi(x) \sim (z'(x))^{-\frac{1}{2}} (z(x))^{\frac{\alpha+1}{2}} \exp(-z(x)/2) L_n^{(\alpha)}(z(x)) . \tag{3.16}$$

Picking the third term on the right hand side of (3.15) as a constant and setting $(z')^2 z^{-1} = C$ we get the LI case [20]. Equation (3.15) can be rewritten as

$$E - V(x) = C\left(n + \frac{\alpha+1}{2}\right) - \frac{C}{4}z(x) - \frac{C}{4z(x)}\left(\alpha + \frac{1}{2}\right)\left(\alpha - \frac{1}{2}\right) . \tag{3.17}$$

According to (2.9) the solution of the defining differential equation of $z(x)$ is given by

$$z(x) = \frac{C}{4}(x + \bar{\delta})^2 \tag{3.18}$$

with $\bar{\delta} = \delta/C^{1/2}$. Again we find that only the $\bar{\delta} = \mathrm{i}\epsilon$ choice with real ϵ can result in a $\mathcal{PT}$ invariant potential and that the $\mathcal{PT}$ transform of $z(x)$ is

$$\mathcal{PT} : z(x) = \frac{C}{4}(x + \mathrm{i}\epsilon)^2 \longrightarrow \tilde{z}(x) = \left(\frac{C}{4}(-x + \mathrm{i}\epsilon)^2\right)^* = \frac{C}{4}(x + \mathrm{i}\epsilon)^2 = z(x) \ . \tag{3.19}$$

The $\mathcal{PT}$ transform of (3.17) is

$$E^* - (V(-x))^* = C\left(n + \frac{\alpha^* + 1}{2}\right) - \frac{C}{4}\tilde{z}(x) - \frac{C}{4\tilde{z}(x)}\left(\alpha^* + \frac{1}{2}\right)\left(\alpha^* - \frac{1}{2}\right) \ . \tag{3.20}$$

Comparing Eqs. (3.17) and (3.20) we find that $\mathcal{PT}$ symmetry holds if $(\alpha^2)^* = \alpha^2$ is satisfied, i.e. if α is purely real or imaginary ($\alpha^* = \pm\alpha$). In the former case the energy eigenvalues will be real, despite the complex potential terms.

Using (3.16) a particular solution of the corresponding Schrödinger equation is written as

$$\psi(x) \sim (z(x))^{\frac{2\alpha+1}{4}} \exp(-z(x)/2) L_n^{(\alpha)}(z(x)) \ . \tag{3.21}$$

Since $\mathrm{Re}(z) > 0$ for $x \to \infty$ (and also for $x \to -\infty$), the solutions vanish asymptotically.

In the conventional treatment of this problem [20] we chose $\bar{\delta} = 0$ in order to obtain a radial problem defined on the positive semi-axis. Then we also had $C = 2\omega > 0$ which rendered the energy to be positive, and $\alpha = l + 1/2$ to account for the centrifugal term. This means that the wave function behaves like x^{l+1} near the origin. Solutions which are non-zero at the origin are not considered physical in the conventional case, when solutions only on the positive semi-axis are taken into account. However, in the $\mathcal{PT}$ symmetric case the singularity represented by the centrifugal barrier vanishes if $\epsilon \neq 0$ holds, therefore the problem can (and should) be extended to the full x axis. In this case the general solution of the problem can be written in terms of two confluent hypergeometric functions [27]:

$$\psi(x) \sim \exp(-z(x)/2)\left[C_1 z^{\frac{2c-1}{4}}(x) F(a, c, z(x)) + C_2 z^{\frac{3-2c}{4}}(x) F(1 + a - c, 2 - c, z(x))\right] \ . \tag{3.22}$$

For $C_2 = 0$, $a = -n$ and $c = \alpha + 1$ (3.22) reduces to (3.21). The solutions with $C_1 = 0$ are now allowed, and this leads to a richer spectrum than that obtained in the conventional treatment. This is because α and $-\alpha$ both become allowed in the formulae. This has been discussed in Ref. [11], where the $\mathcal{PT}$ symmetric harmonic oscillator was introduced, and this possible double sign of α has been attributed to a 'quasi-parity' quantum number. The parameters used there are related to the present ones via $C = 4$, $\alpha^2 - 1/4 = G$ and $\epsilon = -c$.

We note that this extension of the radial problem to the full line in the $\mathcal{PT}$ symmetry context also contains the one-dimensional harmonic oscillator. In that case the centrifugal barrier does not appear, which corresponds to setting α to $1/2$ and $-1/2$. The generalized Laguerre polynomials then reduce to Hermite polynomials, which are odd and even, respectively, corresponding to the odd and even solutions of the one-dimensional problem. In the $\mathcal{PT}$ symmetric context there is no point in discussing the Hermite polynomials and the one-dimensional harmonic oscillator separately, as in Ref. [20] for ordinary quantum mechanics.

When we attempt to analyze the LII (Coulomb) and LIII (Morse) cases in the $\mathcal{PT}$ symmetric context by solving the differential equations $(z')^2 = C$ and $(z')^2 z^{-2} = C$ (as in [20]), we arrive at limits of applicability of the present approach. Their nature is clearly visible from the form of the general solutions (3.16): the normalizability of the wave functions does not depend on the powers of the various terms in (3.16) as for Jacobi polynomials, rather the boundary conditions are determined by the $z(x)$ function itself, which appears in an exponent. In particular, one should have $z \to \infty$ for $x \to \pm\infty$ to secure normalizability of the wave functions. This was guaranteed in the LI (harmonic oscillator case) by the form of $z(x)$ in (3.18), but the corresponding solutions in the Coulomb and the Morse cases, i.e.

$$z(x) = C^{1/2}x + \delta \tag{3.23}$$

and

$$z(x) = \exp(C^{1/2}x + \delta) \tag{3.24}$$

lead to infinities at one limit. In conventional quantum mechanics the latter two cases are considered radial problems, therefore it is enough to have regularity of $z(x)$ for $x \to \infty$. A way around this problem can be found if one replaces the linear integration path $(x + \mathrm{i}\epsilon)$ with curved ones. One possible way to find such curved integration paths is to apply a variable transformation to the $\mathcal{PT}$ symmetric harmonic oscillator problem [11] to get $\mathcal{PT}$ symmetric Morse [16] and Coulomb [18] potentials.

IV. SUMMARY

We revisited a method introduced previously [20] to analyse systematically a large class of potentials and studied the conditions under which $\mathcal{PT}$ symmetric potentials can be derived with it. This method classifies the potentials in terms of the special functions appearing in the solutions of the Schrödinger equation, and also the function $z(x)$ describing the variable transformation taking the Schrödinger equation into the second-order differential equation of some given special function. Here we considered the Jacobi polynomials and the generalized Laguerre polynomials as special functions.

When the method is extended to $\mathcal{PT}$ symmetric problems, some modifications of the original formulation have to be made. One is that a constant of integration representing previously only an unimportant coordinate shift became an important factor, notably in most cases it has to be defined a purely imaginary number in order to get $\mathcal{PT}$ invariance. (Exception for this are some trigonometric potential forms, where specific real components of it can also occur, due to periodicity.) Choosing this constant to be imaginary meant that the $\mathcal{PT}$ transform of the function $z(x)$, $\tilde{z}(x) \equiv (z(-x))^*$ was related to the original $z(x)$ by $\tilde{z}(x) = \pm z(x)$ in most cases, which made it easier to secure $\mathcal{PT}$ symmetry for the potentials.

Besides the imaginary coordinate shift, sometimes certain conditions also had to be imposed on the potential parameters to guarantee $\mathcal{PT}$ symmetry. We showed that one can find purely real energy spectra in all these cases, although the potentials have complex terms in general. This is a natural consequence of the method we followed: as long as the constant C which appears in the defining differential equation of the function $z(x)$ is real, real energy eigenvalues can be obtained. We identified several $\mathcal{PT}$ symmetric problems found previously and interpreted them in terms of our general scheme.

Our approach supplies equations for $E - V(x)$, and we studied the $\mathcal{PT}$ transform of the potential starting from this kind of equations: $\tilde{E} - \tilde{V}(x) = E^* - (V(-x))^*$. Adding to these equations the kinetic term, which is, of course, $\mathcal{PT}$ invariant, the $\mathcal{PT}$ symmetry of the whole Hamiltonian can also be studied.

For $\mathcal{PT}$ symmetric potentials the strong condition of Hermiticity is lost. Such a possibility has been discussed by Andrianov et al [10] who noticed that in supersymmetric quantum mechanics (SUSYQM) the superpotential $W(x)$ can become complex in general, therefore the SUSYQM operators $A = \frac{\mathrm{d}}{\mathrm{d}x} + W(x)$ and $A^{\dagger} = -\frac{\mathrm{d}}{\mathrm{d}x} + W(x)$ can cease to be the adjoints of each other, and give non-Hermitian Hamiltonians $H_- = A^{\dagger}A$ and $H_+ = AA^{\dagger}$. In Refs. [15,8] it was shown that the reality of spectra of certain $\mathcal{PT}$ symmetric potentials can indeed be understood as a direct consequence of their isospectrality with a Hermitian SUSYQM partner. In a next step of development, a consequent understanding of combination of supersymmetry with the genuine $\mathcal{PT}$ symmetry has recently been achieved in Ref. [34]. Within our present approach a straightforward way towards complexified potentials which support real spectra quite naturally follows from the general form of superpotentials

$$W(x) = -\frac{\mathrm{d}}{\mathrm{d}x}\ln f(x) = -\frac{1}{2}Q(z(x))z'(x) + \frac{1}{2}\frac{z''(x)}{z'(x)} \tag{4.1}$$

as studied in Ref. [20].

One important implication of $\mathcal{PT}$ invariance is [2] that the problems which have been interpreted as radial problems and were defined only on the positive half axis $x > 0$ can now be extended to the full x axis, because the terms singular at the origin become finite due to the introduction of the imaginary coordinate shift. Other singularities are also cancelled when the imaginary coordinate shift is implemented. This, of course, relaxes the boundary conditions of the solutions and as a consequence, previously disqualified solutions become acceptable. This mechanism leads to a richer spectrum in general.

We note that the present systematic method of constructing $\mathcal{PT}$ symmetric complex potentials is different from the conventional approaches to complex potentials, when simply some potential parameters are put to complex values, and when the energy eigenvalues also become complex. In our method the complex potential terms that appear due to the $\mathcal{PT}$ symmetry are introduced in a way that can prevent the energy from turning complex. Indeed, the term responsible for the energy in the formulae (see e.g. Eq. (2.6)) is "hand-picked" and can be guaranteed to remain real. This selection also defines the variable transformation function $z(x)$ as in (2.9).

In fact, in a line tested by the orthogonal polynomials the case of the hypergeometric and the confluent hypergeometric functions can also be studied naturally. In Ref. [20] devoted to the Hermitian case it was shown that the method supplies an appropriate framework to identify all the known shape invariant potentials but it makes accessible much wider potential classes too. In particular, the procedure works quite naturally for the Natanzon potentials. The PIII potential [30], for example is derived in the same way as the PI and PII potential classes by solving the $(z')^2 z(1-z^2)^{-2} = C$ differential equation, which turns the last term of (3.1) into a constant (energy) term. The generalized Coulomb problem [31,32], which contains both the Coulomb and the harmonic oscillator potentials as a special case is obtained by setting the combination of two terms in (3.15) to a constant. This is also the case for the Ginocchio potential [28], which can be obtained from (3.1) by setting $\alpha = \beta$, reducing the Jacobi polynomial to an ultraspherical (or Gegenbauer) one [27], and considering the $\phi(z) = (\delta + 1 - z^2)(1 - z^2)^{-2}$ in (2.8) [35].

The "implicit" nature of the latter Hermitian examples might complicate the $\mathcal{PT}$ symmetric formalism to some extent, but in principle, appropriate complexifications can be derived similarly. It can be shown, for example, without deeper analysis that the Ginocchio potential [28] can be made $\mathcal{PT}$ symmetric by using our method. In particular, the actual form of Eq. (2.9) is now [35]

$$\delta^{\frac{1}{2}} \tanh^{-1}\left(z\delta^{\frac{1}{2}}(\delta + 1 - z^2)^{-\frac{1}{2}}\right) + \tan^{-1}\left(z(\delta + 1 - z^2)^{-\frac{1}{2}}\right) = C^{\frac{1}{2}}x + \mathrm{i}\epsilon \ , \tag{4.2}$$

and even this implicit functional form shows (e.g. via a series expansion) that the $\mathcal{PT}$ transform of $z(x)$, $\tilde{z}(x) \equiv (z(-x))^* = -z(x)$, therefore $V(x)$, in which $z(x)$ appears only through $z^2(x)$ [35] must be $\mathcal{PT}$ invariant. (We note that we followed the notation of Ref. [35], because it is closer to the formalism used in this contribution, nevertheless, the original notations of Ref. [28] are readily obtained from it [35].) For potentials beyond the Natanzon class, one has to check each case individually. There the $F(z)$ function can have more general forms, and it is not guaranteed that it satisfies a second-order differential equation as in our approach. However, our experience with the present potentials might help us to understand how $\mathcal{PT}$ invariance and real spectra can arise for these more general potentials too.

APPENDIX

Here we present the essential formulae for the general Natanzon potential class obtained using the methods described in Sec. II. We also show how the corresponding results for the Jacobi and generalized Laguerre polynomials can be obtained from them.

Identifying the hypergeometric function $F(a, b; c; z)$ with $F(z)$ in Eq. (2.2), one obtains $Q(z) = cz^{-1} + (c - a - b - 1)(1 - z)^{-1}$ and $R(z) = -abz^{-1} - ab(1 - z)^{-1}$. From this, the current form of (2.6) is

$$\begin{aligned} E - V(x) = & \frac{z'''(x)}{2z'(x)} - \frac{3}{4}\left(\frac{z''(x)}{z'(x)}\right)^2 \\ & + \frac{(z'(x))^2}{4z^2(x)(z(x) - 1)^2}\left[-(a - b - 1)(a - b + 1)z(x)(z(x) - 1)\right. \\ & \left. - c(c - 2)(1 - z(x)) - (a + b - c - 1)(a + b - c + 1)z(x)\right] \ . \end{aligned} \tag{4.3}$$

With the relation

$$z'(x) \equiv \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{2z(1 - z)}{(\mathcal{R}(z))^{1/2}} \equiv \frac{2z(1 - z)}{[az(z - 1) + c_0(1 - z) + c_1 z]^{1/2}} \tag{4.4}$$

the standard form of the six-parameter Natanzon potential family is obtained [25]. From the six parameters three (a, c_1 and c_0) control the variable transformation $z(x)$, while the remaining three,

$$f=-(a-b-1)(a-b+1)\ ,\quad h_0=-c(c-2)\ ,\quad h_1=-(a+b-c-1)(a+b-c+1) \tag{4.5}$$

determine the potential shape. The general solution then is

$$\psi(x)\sim\mathcal{R}^{1/4}(z(x))(1-z(x))^{\frac{a+b-c}{2}}\Big[A_1 z^{\frac{c-1}{2}}(x)F(a,b;c;z(x)) \\ +A_2 z^{\frac{1-c}{2}}(x)F(b-c+1,a-c+1;2-c;z(x))\Big]\ . \tag{4.6}$$

¿From this usually only one of the particular solutions is regular due to the opposite powers $((c-1)/2$ and $(1-c)/2)$ of $z(x)$ in front of the two functions, and the bound-state solutions are expressed with $A_2=0$.

The Jacobi polynomials can be obtained as special cases of the hypergeometric function in various ways. Using $P_n^{(\alpha,\beta)}(z)\sim F(-n,n+\alpha+\beta+1;\alpha+1;\frac{1-z}{2})$ [27] one can identify the PI and PII type potentials with certain subclasses of the Natanzon potential family. In fact, it turns out that the PI class can be obtained setting $a=-4/C$, $c_1=c_0=0$ in $\mathcal{R}(z)$, the PII class corresponds to the $a=0$, $c_1=c_0=1/C$ choice, while the Ginocchio potential follows from $a=-1/C$, $c_1=\delta/C$, $c_0=0$ [35]. In the original notation of [28], the latter case is $a=(1-\lambda^2)/\lambda^4$ and $c_1=1/\lambda^4$, which reduces to the parameter set of the PI case for $\lambda=0$. In fact, this corresponds to the simple Pöschl–Teller limit of the Ginocchio potential.

Similar formulae can be derived for the Natanzon confluent potentials [26] too. Then for the confluent hypergeometric functions $F(a,c;z)$ one has $Q(z)=cz^{-1}-1$ and $R(z)=-az^{-1}$ and (2.6) becomes

$$E-V(x)=\frac{z'''(x)}{2z'(x)}-\frac{3}{4}\left(\frac{z''(x)}{z'(x)}\right)^2 \\ +\frac{(z'(x))^2}{4z^2(x)}\Big[2(c-2a)z(x)-z^2(x)-c(c-2)\Big]\ . \tag{4.7}$$

Using the notation of Ref. [26] (with $z=h$) and the relation

$$z'(x)\equiv\frac{\mathrm{d}z}{\mathrm{d}x}=\frac{2z}{(\mathcal{R}(z))^{1/2}}\equiv\frac{2z}{[\sigma_2 z^2+\sigma_1 z+c_0 z]^{1/2}} \tag{4.8}$$

the general formula of this potential class is obtained [26]:

$$V(x)=\frac{g_2z^2+g_1z+\eta}{\mathcal{R}(z)}+\frac{\sigma_1 z-\sigma_2 z^2}{\mathcal{R}^2(z)}-\frac{5\sigma_1^2-4\sigma_2 c_0}{\mathcal{R}^3(z)}\ . \tag{4.9}$$

Then the energy eigenvalues are obtained by choosing the parameters in such a way that the

$$E=-\frac{1}{\mathcal{R}(z)}\Big[(1-g_2)z^2+[2(2a-c)-g_1]+[(c-1)^2-\eta]\Big] \tag{4.10}$$

becomes a constant. Again, from the six parameters three (σ_2, σ_1 and c_0) control the variable transformation $z(x)$, while the remaining three, (g_2, g_1 and η) determine the potential shape. The general solution then is

$$\psi(x) \sim \mathcal{R}^{1/4}\exp(-z(x)/2)\Big[A_1 z^{\frac{c-1}{2}}(x)F(a,c;z(x))$$
$$+A_2 z^{\frac{1-c}{2}}(x)F(1+a-c,2-c;z(x))\Big] \ . \tag{4.11}$$

The generalized Laguerre polynomials are obtained by $L_n^{(\alpha)}(z) \sim F(-n,\alpha+1;z)$ [27]. The three basic problems discussed in Sec. III follow from these formulae by setting $\sigma_1 = 2/C^{1/2}$, $\sigma_2 = c_0 = 0$ (LI case, harmonic oscillator), $\sigma_2 = 2/C_n^{1/2}$, $\sigma_1 = c_0 = 0$ (LII case, Coulomb problem) and $c_0 = 2/C^{1/2}$, $\sigma_1 = \sigma_2 = 0$ (LIII case, Morse potential).

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